The
Place of Mathematics
in
Secondary Education

The National Council of Teachers of Mathematics

FIFTEENTH YEARBOOK

The Place of Mathematics *in* Secondary Education

THE FINAL REPORT OF
THE JOINT COMMISSION OF
THE MATHEMATICAL ASSOCIATION
OF AMERICA AND
THE NATIONAL COUNCIL OF
TEACHERS OF
MATHEMATICS

BUREAU OF PUBLICATIONS
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EDITOR'S PREFACE

THE present Yearbook is the final report of the Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics on "The Place of Mathematics in Secondary Schools" As editor of the series of Yearbooks I wish to express my personal appreciation both to the Mathematical Association of America and to the National Council of Teachers of Mathematics for the generous way in which they have cooperated to make this Yearbook possible.

W. D. REEVE

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PREFACE

THE Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics, on the Place of Mathematics in the Secondary Schools, was organized in 1935 to take over the work of separate committees of the two organizations that had been appointed some months earlier to study the problems of secondary mathematics During the first two years of its existence the Commission was able to hold conferences only in connection with meetings of the sponsoring societies, and not much progress was made toward the preparation of a report[1] In January, 1937, the General Education Board generously made a grant to the Commission of $5000, an amount subsequently increased to $6500, and this subsidy made it possible for the Commission to hold meetings and carry on its work more effectively.

The Commission had been left free to choose its own approach to the general problem of mathematics in secondary education. It became apparent very soon, however, that the Commission would not be able to consider all phases of such a broad subject nor treat exhaustively the topics it selected for discussion. Among the primary difficulties in preparing the Report was that of achieving a proper balance between general principles involved and specific proposals concerning methods of developing these principles The Commission believed that there are three groups of persons especially interested in such a report: (1) students of education, (2) administrators, (3) classroom teachers of mathematics. Obviously these three groups will view

[1] A meeting of the original committee of the Mathematical Association in December, 1934, was made possible by funds furnished by the Commission on the Secondary School Curriculum of the Progressive Education Association, of which Dr V T. Thayer was chairman

the Report from somewhat different standpoints. Thus the student of education and the administrator may be interested chiefly in different phases of the broader aspects of the problem, while the teacher may be seeking answers to specific questions of instruction. It was in regard to possible interests of the teacher that restrictions in the scope of the Report had especially to be made. The Commission could not prepare a general handbook, and it did not seem advisable to discuss methods of instruction.

Within the Report itself the Commission has constantly sought to make it clear that the members recognize that there is no one perfect pattern of instruction in mathematics. Continual experimentation is necessary if mathematical teaching is to meet the demands of changing school conditions, and it would be unwise to attempt to mold mathematics into a single shape, or to make any plan too rigid. Good programs departing considerably from those that are set forth in this Report are being successfully carried on. However, the Commission does believe that the programs suggested are sound and, moreover, sufficiently flexible to meet a wide variety of needs if they are intelligently interpreted and followed

If there is marked difference of opinion among teachers concerning the most satisfactory curricula for normal pupils, there is still more disagreement concerning programs for slow pupils, and, to a less degree, concerning those for superior pupils Although it is believed by many that only a start has been made toward the investigation of these problems, the Commission felt that it could not evade questions that are so pressing at the present time It is hoped that the discussion of general principles and the specific suggestions that are given will aid those schools and those teachers who have not already found a satisfactory solution to the problem.

The Report is not burdened with extensive references, as is so often the case in educational discussions. Views expressed are those held by the members of this Commission, based on their

own experience, their reading and study, and their discussions with other persons It seemed that little would be gained by trying to trace ideas to doubtful sources or by the citation of writers who are known to concur with the statements in the Report When other writers are quoted, they are usually quoted not because they are regarded as unimpeachable authorities behind whom the Commission can take shelter, but because they have given forceful or felicitous expression to views that the Commission endorses. On the other hand, the two basic programs for mathematics instruction that are set forth in Chapters V and VI have elements that are so familiar it seemed unnecessary to support them by references. In some chapters it seemed desirable, however, to give short bibliographies, and it is hoped that the references included will prove sufficient for the reader who desires a further discussion than that given in the Report.

Several of the chapters were published in 1938 in preliminary form in two pamphlets in order to receive criticisms and suggestions Many letters were received from individuals as well as more or less extensive comments from committees and groups that had discussed the preliminary Report. All of this material was very valuable in the preparation of the final Report. It is not possible to record here the names of all the many persons who, by giving aid and counsel, have demonstrated their interest in the work of the Commission, but the Commission desires to express its sincere appreciation to all of them. The obligation of the Commission to the following persons is especially great: Professor Harl R. Douglass of the University of North Carolina, who met with the Commission during one of its sessions; Professor E. R. Breslich of the University of Chicago and Professor Virgil S. Mallory of Montclair State Teachers College, who in addition to furnishing criticisms of the preliminary Report read some of the revised chapters and responded generously to questions that were asked them; Dr. C. A. Atherton of Hershey Junior College and Dr. R. J Hannelly of Phoenix

Junior College, who helped in the revision of the chapter on the Junior College No responsibility should be placed upon these persons by anyone who disagrees with the Report. Although it gave careful consideration to advice and criticism, the Commission did not always follow the recommendations made, so that the persons mentioned will themselves not concur with all that is said in the parts of the Report they especially scrutinized; but the Report was undoubtedly improved by their counsel and their suggestions. Finally, the Commission wishes also to record its great indebtedness to the General Education Board for the financial assistance it gave.

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"I had been to school most all the time and could spell and read and write just a little, and could say the multiplication table up to six times seven is thirty-five, and I don't reckon I could ever get any further than that if I was to live forever. I don't take no stock in mathematics anyway."

—HUCK FINN

INTRODUCTION

THE ROLE OF MATHEMATICS IN CIVILIZATION

"There was things which he stretched, but mainly he told the truth."

—THE ADVENTURES OF HUCKLEBERRY FINN

As a background for discussing the place of mathematics in any system of education, it seems appropriate to give some attention to the ways in which this subject has assisted in the development of our civilization and the extent to which we now depend upon it. So old are some of the uses of mathematics and so deeply are they embedded in our culture that they are in danger of being taken for granted and forgotten. On the other hand, at the present time the subject is finding employment in new ways that may not be known to those who are not directly concerned with mathematics, or even to some teachers of the subject. Accordingly the theme that appears in the caption above is here developed briefly.

HISTORICAL PERSPECTIVE

Some of our oldest historical documents have to do, either exclusively or incidentally, with mathematics. The beginnings of the subject therefore belong definitely to pre-history. Though present-day studies are revealing the existence of a much greater knowledge of mathematics in antiquity than we had supposed, we shall never be able to uncover the first steps in its development any more than we can know in any satisfying way about man's first efforts toward artistic expression. It was a realization of the age-old character of mathematics and the social causes of its origin that led Professor Hogben to give the picturesque

summary, "The history of mathematics is the mirror of civilization."[1] During its centuries of growth the subject has received contributions from different peoples, has been a common heritage for much of mankind, and throughout the civilized world is regarded as indispensable for further progress. The realization that so much mathematics has come down to us from past years often leads to the belief that here at least is one science which has completed its growth Such a view is, however, quite erroneous, for the subject is one which today is growing so rapidly, as the great number of research papers published here and abroad indicate, that the present has been described as the Golden Age of Mathematics

MATHEMATICS AN OUTGROWTH OF FUNDAMENTAL HUMAN NEEDS

However refined and abstract much of mathematics now is, one may be sure that its origin had to do with the commonplace matters of food, clothing, and shelter Questions inevitable for all human beings—such as how many? how much? how long?—could be answered only by *counting* and *measuring* These two processes, under the relentless stimulus of hunger, cold, and desire, gave the basic impulse toward the creation of mathematics, and today, in spite of its growth and ramifications, *number* and *form* remain among the fundamentals of the subject

ARITHMETIC AND GEOMETRY

Having received instruction in numbers during early childhood, people take them for granted and so do not realize that the development of the number system was an epic achievement. One is more likely to wonder how it ever was possible to get along with horses and buggies than to wonder how the Romans managed with their clumsy numerical notation. But upon reflection one sees that the ideas, the processes, and the applica-

[1] Hogben, Lancelot. *Mathematics for the Million*, p. 32. W. W. Norton and Co., 1937

tions of *arithmetic*, as we have it today, form a highly perfected, permanent, and universal social heritage.

The material world which surrounds us reveals form as clearly as it suggests number Although natural objects present themselves in countless varieties, certain forms have a tendency to reappear, and this constant recurrence suggests basic concepts and stimulates the creation of an appropriate vocabulary Circles, for example, are repeatedly forcing themselves upon us; one sees them in the sun, the full moon, the ripples on a pond, and everywhere, albeit imperfectly, in the trunks of trees and the stems of plants For centuries the night time sky was regarded as a hemisphere, with the stars fixed upon it, and the modern astronomer, though he knows better, preserves the illusion and still speaks of the "celestial sphere" Long before man had constructed a circular arch he had seen one flash forth mysteriously in the rainbow Various practical arts, forced upon man in the struggle to survive, and raised later to a higher plane by his desire to make living more comfortable and refined, prepared the way for *geometry*

THE GREEK ACHIEVEMENT

The much repeated story that geometry as a deductive science, as distinct from a compilation of miscellaneous rules, was started by Thales after he returned from a business trip to Egypt may not be true There can, however, be no doubt that Thales and other Greeks developed ideas that were to prove revolutionary by laying the foundation for some of the most significant aspects of Western civilization To them belongs the credit of initiating *rational science* Long before the time of the Greeks, as early perhaps as the Stone Age, men possessed certain scraps of scientific knowledge of a practical kind, born heuristically from their experiences Thus they understood something about the use of the lever as an aid in moving heavy objects. At a later period we find quite remarkable technical knowledge in the civilizations of the Chaldeans and the Egyptians. But peoples even as

advanced as these did not succeed in giving a reasoned explanation of natural phenomena and technical processes; apparently they hardly attempted to do so. On the other hand this was precisely what the Greeks did, and with them science as we think of it had its beginnings. Much of the Greek success is due to the development given to mathematics.

OTHER BRANCHES OF CLASSICAL MATHEMATICS

Such concepts as those of equality, congruence, similarity, symmetry, ratio and proportion, woven so firmly into geometry, also contributed, under the spur of problems of indirect measurement, to the development of trigonometry, which had been used by the Greeks mainly in connection with astronomy. As one contemplates the table of chords that Ptolemy gives in his great work, which the Arabs renamed *Almagest* as an act of reverence, he realizes how essential all such work was if man were ever to "control his environment."

The problems that were being subjected to mathematical analysis were constantly becoming more intricate, and revealed the necessity for methods more general and techniques more powerful and universal than the old ones. Out of this need came algebra, a subject which has shown a disposition to crowd into more and more places with benevolent willingness to take upon itself a larger amount of the work of the world.

Appreciating this characteristic of algebra, Fermat and Descartes made a fusion of it and geometry, giving us the study we know as *analytic geometry*. Here a strange thing happened in our outlook upon things. Though to our sight the world looks convincingly like a three-dimensional affair, the new mathematics enables us to deal with a world not so "cabin'd, cribb'd, confined." Adventures into regions of more than three dimensions have been successfully made, the explorers bringing back valuable suggestions for dealing with the physical world, which shows the folly of ever committing ourselves to a policy which considers only things patently "practical."

Soon after algebra and geometry were united, the growing desire to deal effectively with phenomena and problems which had defied analysis brought into the foreground questions about *rates of change* Although certain Greeks, notably Archimedes and Eudoxus, had employed some of the conceptions of the *calculus,* it remained for the inventive and penetrating minds of Newton and Leibnitz to fix the foundations in a definite way The mathematician is justified in growing enthusiastic about this subject, since a new era for mankind began with its discovery, and human culture will never cease to draw heavily upon its striking methods of analysis. When, along with his neighbors, a mathematician observes publicizing of the fact that the world constantly changes, he may be pardoned for some surprise, for he is likely to recall that two and a half centuries ago Newton was talking about fluxions and fluents.

THE SERVICE VALUES OF MATHEMATICS

Since much of our basic mathematics grew out of needs, its high utility should occasion no surprise. As the subject expanded it became more and more self-contained Mathematicians long have regarded it as one of man's most appropriate activities, and they find satisfaction and interest in the study itself This attitude has given an even greater incentive for pressing its development than has the goad of need; nevertheless contacts with the physical world and applications to it have been discovered as by-products, such results being constantly taken up and cultivated by those engaged in various technical pursuits As there is no danger of our being deprived of the results of mathematical inquiries, people may not keep in mind the many ways in which the subject ministers to us and may even forget how primitive our civilization would be without it. Most people, for example, do not know that quite as much use is made of differential equations as of wire in radio work. An inspection of textbooks having to do with numerous technical fields, professions, and trades reveals their great dependence

upon mathematics, and, if made thoughtfully, can well be recommended for those who aspire to be competent appraisers of our culture. (See Appendix I.)

THE NEWER USES

Since the beginning of the century the science of physics has been greatly changed. The new work not only has employed novel apparatus but has required more advanced mathematics. In the case of the general theory of relativity, mathematics has been used in a more searching way than ever before, since purely mathematical attributes became a guide in the quest for physical laws The greater mathematizing of physics was to be expected It was merely another step in an evolution long under way More significant is the fact that some studies, until recently not mathematical at all, are rapidly becoming so. Perhaps the most notable example is chemistry, which is no longer almost solely a matter of test tubes and crucibles plus a little arithmetic, but one of differential equations and integral signs.[2] Mathematics is a means not only for giving precise and economical statements for truths previously expressed less accurately and gracefully but also for affording new methods of discovery,[3] in some instances excelling the resources of the laboratory, just because it can bring one to grips with basic considera-

[2] So striking has been the change that one chemist has remarked "Chemistry has graduated from the class of the descriptive sciences into the class of the exact sciences, and has taken its place by the side of physics and engineering as a branch of mathematics." Daniels, Farrington "Mathematics for Students of Chemistry" *The American Mathematical Monthly*, Vol. XXXV, pp 3 9, 1928

[3] An outstanding instance of what mathematics has done for chemistry with benefit to the entire world has been described as follows·

"A modest professor of mathematical physics at Yale University discovered the phase rule and other thermodynamical laws of heterogeneous solutions, and today these laws are the very cornerstone of petroleum refining and of other chemical industries It would not be inappropriate if the oil barons and motor magnates celebrated the birthday of Willard Gibbs as the most important anniversary in their calendar—for the billions of gallons of gasoline which last year activated the millions of automobile motors were distilled,' not only out of the refinery stills, but also out of the mathematical equations of the meditative professor." Gray, George S "Science and Profits." *Harpers Magazine*, April, 1936.

tions In psychology and physiology, advances of new kinds are made through the application of mathematics, while certain types of geological studies can be made only by means of mathematics, meteorology being quite saturated with it

Mathematical methods have appeared somewhat slowly in the social studies, as was to be expected on account of the traditional descriptive qualities of these subjects The use of mathematics in economics will be considered briefly. Although in courses in economics there is reference to "laws" and "principles," from which it is asserted that conclusions are derived by deductive processes, mathematics has often been avoided. At most, a "curve of supply" and a "curve of demand" may have been drawn, and their intersection pointed to with a certain amount of satisfaction Economists acquainted with mathematical methods have frequently felt a little uncomfortable at the inadequacy of the traditional procedures, a notable example being Marshall, who even in the first edition of his great work, so often reprinted and revised, made use of the calculus only in the appendix out of consideration probably for his readers. There was no lack of emphasis, however, in the view which he expressed that economics must be developed by mathematics if it is to arrive at clarity in its basic considerations

It should be observed that it is just one hundred years since the appearance of Cournot's famous *Researches on the Mathematical Principles of the Theory of Wealth* Cournot was a mathematician who sought to aid scholars in other fields, and his example has been followed by later mathematicians who have attempted to show in some detail wherein may lie ways of progress in the socially important subject of economics [4]

[4] A bibliography of mathematical economics from 1711 to 1897, by Irving Fisher, will be found in the translation by N. T Bacon of Cournot's work, The Macmillan Co , 1929 Important recent works on the subject by American mathematicians are Evans, G C., *Mathematical Introduction to Economics*, McGraw-Hill Co., 1930, Roos, Charles F , *Dynamic Economics*, The Principia Press, 1934. Reference may also be made to the works of such economists as H L. Moore and Henry Schultz The increasing use of mathematics in economics is also shown by the establishing of the journal *Econometrica*.

The mathematics that is of great use in developing concepts and relationships in the field of economics is not solely the mathematics of statistical analysis, but mathematics of a more fundamental and classic form. It is of course true that statistical methods are of growing importance in the study of economics as elsewhere From its very modest estate of a few years ago, the theory of statistics has grown significantly and has been successfully employed, for instance, in education and sociology. Statistical competence can now be regarded as a necessary qualification for many activities.

MATHEMATICS AS A MODE OF THINKING

It has already been suggested that though mathematics came into being through quite elemental needs, it has advanced far beyond that state. By the time geometry had attained the perfection that the Greeks gave to it, mathematics had emerged as an instance of rigorous deductive thinking. Since then the word has denoted something over and above its theorems and its results To *mathematize* a subject does not mean merely to introduce into it equations and formulas, but rather to mold and fuse it into a coherent whole, with its postulates and assumptions clearly recognized, its definitions faultlessly drawn, and its conclusions scrupulously exact. Mathematical methods set up an ideal, a norm, a pattern, which though not attainable in all the activities necessary and proper to man, serve as a relentless spur to improvement. Mathematics succeeds because it searches tirelessly for the principles that underlie a situation or the weak point in a problem It analyzes, it dissects, and it relates, it tends to eliminate what is irrelevant or superfluous in order to attain an economy in thinking and expression

THE SIGNIFICANCE OF ACCURATE THINKING

At a time when so much is heard of social needs, it is well to emphasize those traits which distinguish men from other creatures. Assuredly it is not the social instinct, for there are animals

and insects which have highly organized social lives, nor is it even such qualities as devotion and sacrifice. But man alone has the language sense and the high degree of reasoning power which allow the progressive development of one generation after another, by recording, analyzing, and using past experience. A squirrel, like man, must deal with the problem of the normal granary, but it does not concern itself with questions of abstract reasoning, which man cannot escape.

Man sets ideals for the things he does, and accordingly an ideal must be set for thinking. If accuracy, cogency, should be this ideal, then it is attained in mathematics. That is why mathematicians are little troubled by the question whether they see possible "utility" in this or that, but feel that they are engaged in the most *human* of enterprises, accurate thinking, and are not demanding immediate outcomes.

Can man by reasoning arrive at conclusions that represent his highest conception of truth? The answer is yes, and it is the constant business of mathematics to show this. If one wishes an example, he may recall the proof that n^3-n is divisible by 6 for all integral values of n. Is rational thinking "profitable" as well as possible? One need merely recall what the modern world owes to mechanics in order to obtain the answer.

The fact that in many situations there are variable and imponderable factors which influence people, in addition to emotional prejudice, has led to the claim that we should be more concerned with rational living than with rational thinking. It seems quite impossible to separate rational living from rational thinking; perhaps the difference between the two is one of degree rather than one of fundamental qualities. In any given situation we may either act one way or another or refuse to act; but in any case, we are held responsible for our decision and we must face the consequences of it, whether we like to do so or not. Surely prudence suggests that we seek to predict on the basis of logical analysis what the consequences of a given mode of behavior will be.

MATHEMATICS AS RELATIONAL THINKING

Professor C J. Keyser has made the following forceful statement in regard to relational thinking:

> Each thing in the world has named or unnamed relations to everything else. Relations are infinite in number and kind. *To be is to be related.* It is evident that the understanding of relations is a major concern of all men and women. Are relations a concern of mathematics? They are so much its concern that mathematics is sometimes defined to be the science of relations.[5]

It is precisely in this fundamental problem of searching for relationships and giving accurate expression to them that mathematics has been so successful. Though the thought of relationship was present in parts of Euclid and in early trigonometry, the general quest was handicapped until appropriate symbolism came into existence, as it began to do with the appearance of algebra The letters of the alphabet, having been freed of the task of representing special numbers through the development of the Hindu-Arabic notation, became available for all numbers, and mathematics began to reveal the great power of symbolic thinking With the development of calculus came an entirely new category of relationships, namely those in which change is involved Equations in mathematics, from the simple formulas for the area of a rectangle or a circle to the differential equation for a radio circuit or the integral equations which appear in mathematical economics, are nothing more than expressions of the ways in which one variable is related to others Similarly, all the tables we now have for elementary and higher functions are merely tables showing very explicitly the way in which certain quantities are connected As mathematics grows it constantly acquires and develops ways of dealing with relationships which before had eluded it. Because a good deal of effort is needed to understand these methods, there has been a tendency for workers in some sciences to neglect them and thereby throw upon

[5] Keyser, C J *Mole Philosophy and Other Essays*, pp. 94-95 E P Dutton and Co, 1927

mathematicians the responsibility not only of elaborating the methods but also of showing the applications

INFLUENCE OF MATHEMATICS ON OUR PHILOSOPHICAL OUTLOOK

The success that has attended the development of mathematics and the benefits that have come through its wide employment have had an important part in enlarging the general intellectual background of the present day. Certain philosophers, for example Kant, have drawn very heavily upon mathematics; but it is hard to estimate how much influence such highly refined theories of knowledge have had outside the small circle that gives attention to metaphysical considerations, though undoubtedly some influence has been exerted by those who read Kant or by those who read those who themselves had read Kant The reality of mathematical influence, however, is far clearer than such a debatable example indicates, for mathematics, having been studied by the majority of well-educated people, has acted directly upon them, so there is not the need of the interposition of the erudite philosopher.

Even people who have no occasion to "use" more than the elements of mathematics and who have not followed its advances regard it as a kind of stabilizing influence in a world quite uncertain about itself, and are heartened by the reflection that over an ever-widening area of activities man's mind can work unerringly, without passion, prejudice, or selfishness. Of course they do not take this as meaning that all our problems may be brought within the compass of mathematical methods or that extensive mathematical training is desirable for every enterprise. The limitations of mathematics are to be conceded quite as candidly as one admits those, for example, of music, of painting, and of sculpture. But just as these fine arts refresh and give renewed spirit, so mathematics braces our mental background by revealing the fact that within some subjects it is possible to achieve the highest ideals of thinking. The realiza-

tion that this is so is one of the things that should increasingly condition life today.

Our literature, both that which is read and enjoyed over a long period of years and that which is definitely ephemeral, reveals the ideas just set forth. Few books have received as high praise as Wordsworth gave to Euclid in the lines:

> The one that held acquaintance with the stars,
> And wedded soul to soul in purest bond
> Of reason, undisturbed by space or time.

It is clear that Wordsworth recognized a very great human significance in geometry and saw a value in its theorems quite superior to what is meant by utility A social philosophy is set forth in the lines above, one that regards men and women as being brought together by their intellectual possessions. Whether such a community of interests will ever be achieved to a significant extent is not the question, the important thing being that an ideal, drawn from Euclid, is set before us in language so felicitous that one turns back again and again to read and reflect. Such ideals make our civilization seem better and richer, especially when they are set forth in words that tend to strengthen the ideals themselves

Perhaps even more haunting is the line of Edna St. Vincent Millay, "Euclid alone has looked on Beauty bare," and the last part of her sonnet expresses a thankfulness for hours spent with geometry that must stir all who read our language

> O blinding hour, O holy, terrible day
> When first the shaft into his vision shone
> Of light anatomized. Euclid alone
> Has looked on Beauty bare Fortunate they
> Who, though once only and then but far away
> Have heard her massive sandal set on stone [6]

Algebra, too, has left an imprint of its power and gracefulness that remains fresh in mature years, as shown recently by the novelist James Hilton when he linked music and pure mathe-

[6] From *The Harp-Weaver and Other Poems*, published by Harper and Brothers. Copyright 1920, 1921, 1922, 1923, by Edna St Vincent Millay.

matics together, asserting that, in the long run, it is something like the binomial theorem or a Bach fugue that counts

Turning to the daily press we find the same theme Reporting upon the Harvard Tercentenary, William L Laurence wrote for the New York *Times* of September 6, 1936:

> The sessions this week dealt with the achievements of man's mind in the realms of the abstract—mathematics, astronomy, astrophysics, and cosmogony—with world leaders in these fields recounting their latest adventures
>
> Beginning with the most abstruse mathematics, calculating dimensions of the universe, the conference proceeded to the more concrete sciences dealing with the structure and constitution of man's cosmos
>
> Thus the conference went from the pure achievements of man's intellect outward to the stars and galaxies The general picture produced was that of man at his best, elevated by the superiority of his brain power

It is not likely that Mr. Laurence would have written in this fashion, or that his paper would have printed what he wrote, if it were not realized that there are people who take inspiration from dwelling a little on man's intellectual achievements. Confused as people are over current problems it is well to remind them that there is indeed such a thing as "man at his best."

We have come a long way in our thoughts, from man the primitive creature, learning to count in order to cope better with a hostile world, to the lines of Wordsworth that take one's thoughts into space, as well as inward to our intellectual possessions, to Miss Millay's conception of beauty and her deeply felt gratitude to geometry, to Mr. Hilton's lingering memory of the grip of algebra, and Mr Laurence's suggestive lines on man at his best Surely any question of whether mathematics influences our philosophy is answered

CHAPTER I

LOOKING AT MODERN EDUCATION AND ITS GENERAL AIMS

"Perhaps after censuring all the opinions that have been put forward on this obscure subject, one ought to propose some theory of one's own"

—HERODOTUS

OUR FAITH IN EDUCATION

Education in a Democracy. It is clearly the desire of the people of the United States to give wide educational opportunities to boys and girls From the founding of the Republic to the present, men and women prominent in the national life have maintained that the successful democracy which we cherish as an ideal must rest upon the proper and widespread education of its people It has been held not only that democracy might cease to exist without education, but also that a democratic order, even if apparently successfully achieved, must seek constantly to extend the advantages of education to those who do not as yet enjoy them Thus education is both a primary means to an end and an end in itself never to be regarded as completely attained.

A Story of Rapid Growth. The growth of our school system shows how vital and powerful the educational impulse has been, for schoolhouses and school children are unquestioned realities. With the increase of numbers in the schools there have arisen grave problems for teachers and administrators. Especially has this been true in the secondary schools, the rapid growth of which is shown by impressive figures frequently quoted.[1] The

[1] The number of pupils has doubled every ten years since 1890. In 1936 more than 6,000,000 of the approximately 10,000,000 youth aged fourteen to seventeen were in high school Douglass, Harl R *Secondary Education for Youth in*

newcomers in the high school have been from social and economic groups whose members previously had seldom gone beyond the elementary school, if indeed that was completed. Various reasons have been suggested to explain the influx. Perhaps one reason is that a high school education was considered a means to increased social prestige, while at the same time technological developments in industry made it more difficult for boys and girls of high school age to obtain employment Industry now requires more technical knowledge or sufficient skill and endurance to perform routine tasks rapidly for hours at a stretch. Whatever the reasons may be, however, it remains that the high school population has greatly increased, a fact indicating that the people as a whole have faith in education

NEED OF A MORE COMPREHENSIVE VIEW OF EDUCATION

In the past, the policy of the school was simply that of determining more or less arbitrarily what the boys and the girls should do Traditionally, schools have required certain subjects for study and have provided other activities in which pupils were expected to engage. But with changing pupil personnel, traditional curricula proved unsuited to many newcomers, who seemed to derive little benefit from their study The old studies were therefore modified, and new ones were provided with the hope of meeting the changed conditions. But these measures did not meet the situation fully, and it has come to be recognized that the problems involved must be considered in a more fundamental way.

Some of the efforts that have been made to solve the challenging problems will be briefly described.

For as long as two decades attention has been devoted to the development of specific objectives for each of the subject matter fields By making clear the way in which portions of a subject

Modern America, p 28 American Council of Education, Washington, 1937; with statistics from bulletins of the U S Office of Education.

can contribute to achieving the goal or purpose of the whole, the ineffectiveness and aimlessness of much teaching has been reduced. Educational psychologists on their part have sought to improve instruction by indicating various "bonds" that are to be mastered by drill or repetition. In order to have more complete appraisals of pupil progress, testing experts have constructed various kinds of tests—inventory tests, diagnostic tests, and achievement tests—many of which have become standardized and are widely used.

A frank discussion of the educational situation must note that the movements described have, along with their good effects, added some confusion, for there are cases where they have been carried to an extreme. Thus, if a list of objectives is developed to a too great degree of refinement, an actual handicap can be imposed upon good teachers. When one affirms the existence of this fault, he is not denying that there are still many teachers whose work suffers badly from incoherence and the lack of definite objectives On the other hand, tests have sometimes been applied without proper regard for class size, classroom methods, and time allotments Furthermore, in their use consideration has not always been given to the backgrounds of the pupils It is probable that a too highly mechanistic procedure is partly responsible for the emergence of the philosophy of the child-centered school and other pronounced forms of progressive education

The *Eleventh Yearbook* of the National Council of Teachers of Mathematics includes a brief survey of the principal attempts at curriculum revision made during the past few decades Such studies show that there has been an ever-increasing expansion of the scope of secondary education, in the interest of both enrichment and greater flexibility They also show that there has been a trend away from authoritarian prescriptions, and that this tendency to remove fixed requirements has led to a marked reduction of emphasis on definitely formulated "subject matter courses." Some educators now propose extensive participation

in curriculum making on the part of classroom teachers and pupils. The curriculum is then viewed as "emerging" from day to day, in accordance with ever new plans of "reconstruction," "adaptation," "integration," and the like. But it is not clear how such a program can provide for continuity or coherence, or, perhaps, even avoid some degree of educational chaos.

The recognition that something more is needed than improvement of instruction through better utilization of pupil psychology, construction of tests for achievement, and analysis of objectives for specific subjects has led to the proposal of what are considered to be more fundamental approaches to educational problems It is argued that the schools have been employing quite inadequate or erroneous guiding ideals and principles, though the acknowledged benefits which have accrued from the schools as organized give evidence to the contrary. However, the efforts to find new and sufficient principles or approaches have led to so many proposals that the situation has been made more bewildering to many teachers instead of being convincingly clarified.[2] The question arises as to how the individual teacher or administrator is to choose among such a varied collection of theories and proposals.

Clearly there is need for a comprehensive point of view that will look at education as a whole Only in the light of an inclusive orientation will it be possible to appraise adequately the possible contribution of any subject matter field It is necessary to elaborate dependable criteria of appraisal, of selection, of organization, of progress, and so on

THE DUAL ASPECT OF THE CURRICULUM

Only too often the fact is overlooked that every school activity of importance has both an impersonal and a personal aspect.

[2] A long list of different "approaches" advocated at the present time has been given by Bruner, Herbert B et al, *A Tentative List of Approaches to Curriculum and Course of Study Construction* Mimeographed, Teachers College, Columbia University, Curriculum Library, 1934 The work of Bruner has been used by Norton, John K and Norton, Margaret A, *Foundations of Curriculum Building*, Ginn and Co, 1936 See especially Chapter III

The first has to do with the accumulated or developing experience of the race and is relatively independent of personal opinions or backgrounds. The second has to do with the reaction of the individual pupil or school to this reservoir of information or training. And so, the work of the school has both a static and a dynamic character. On the one hand, the school transmits important cultural possessions of the race, without which our institutions could not go on The maintenance of this heritage gives continuity and distinction to our national life. On the other hand, each generation necessarily re-examines this heritage and adapts it to its own uses in the light of new and changing experiences. Therefore it is inevitable that in many areas of human experience there will be factors that gradually become obsolete while others suddenly assume a crucial importance. The drama of human evolution is thus a continuous struggle between the old and the new. This fact is inevitably reflected in the problems of the curriculum.

Throughout the whole history of education there have been countless attempts to define its functions, to clarify the meaning of the educational process, and to set up programs or regulations for the guidance of pupils and teachers. By this time we should realize that there can be no finality about such definitions or prescriptions Each epoch will insist on interpreting in its own way the educational needs of the rising generation, in accordance with new demands and altered perspectives Since human evolution is an unending process, it follows that education involves the necessity of constant readjustment.

There are, however, weighty stabilizing factors that prevent a complete break with the past. There are, in fact, permanent backgrounds, perspectives, and values so universal and all-embracing that they might be expected to be held continuously in view They are connected with certain permanent physical, mental, social, and spiritual types of human needs, and with elements of our world-wide environment that change only imperceptibly, if at all. It is the function of the school to stress, first

of all, adaptations to permanent features of existence, without neglecting adjustments demanded by each successive age. If there is nothing of lasting significance in the programs of our schools, then all is writ in water and we are indulging in a futile game of self-deception at the expense of the youth of the race. There is wisdom in clinging to a firm belief in enduring backgrounds, and even the revolutionary developments of the recent past, while profoundly affecting human welfare, serve only to strengthen this conviction

ENDURING EDUCATIONAL CONCERNS

Among the human needs that will never lose their importance are those of food, clothing, and shelter. A knowledge of facts and activities centering around these fundamental human requirements should be a concern of the schools. Security, reduction of anxiety, of disease, and of useless drudgery depend to an ever-increasing extent on the study and control of nature and its resources Hence science, with its countless ramifications, will permanently engage the attention of the school For similar reasons, the study of man's social institutions cannot safely be neglected. Again, the practical arts, in conjunction with science, leading by slow degrees to advanced forms of technology, were among man's chief tools in his slow ascent from savagery to the power age in which we now live. Hence, training which aims to develop creative ability in the fields which have to do with the transformation of raw materials into useful and artistic objects, is a vast though still neglected domain in the schools. And in the evolution of science and technology, mathematics has constantly furnished essential and invaluable aid For that reason alone the study of mathematics must be regarded as a permanent ingredient of every balanced school curriculum The fine arts, including music and literature, have provided elements of release from drudgery and sources of appreciation that are among humanity's noblest treasures To furnish continuous contacts with these great and enduring products of human genius would

seem to be a sacred obligation of the school. Situated at a still higher level there is a reservoir of spiritual values and visions pertaining to a nobler and finer life, to ideals of heroism and self-sacrificing endeavor. To be sure, such values may not be suitable for incorporation in textbooks and courses of study. But it is not too much to expect that the school shall, by its spirit and in its daily activities, keep in mind the infinitely subtle and permanent problem of personality growth.

This brief sketch of the school's permanent interests could easily be extended or be made more explicit. Enough has been said, however, to suggest the major aims or objectives of the school But aims or objectives are not cold abstractions to be realized merely by a series of methodically planned activities. They are, rather, the school's confession of faith and an indication of the manner in which it interprets its stewardship.

CHAPTER II

GENERAL OBJECTIVES FOR SECONDARY EDUCATION

"I pray thee overname them, and as thou namest them I will describe them, and according to my description, level at my affection"

—PORTIA IN THE MERCHANT OF VENICE

IN THE preceding chapter the thesis was developed that the school should give adequate attention to the broad areas of experience that have been shown to be of almost universal occurrence and importance. Unless he is brought into contact with them, the pupil will be unable to interpret, to appreciate, and to participate helpfully in important domains of modern life. In the present chapter a somewhat detailed discussion will be given of certain broad and general objectives that should guide the instruction and the training which secondary schools provide for pupils

A Classification of Objectives. It is evident that educational objectives in the last analysis will center around three permanent factors, namely, *the physical universe, society,* and *the child* They are invariant frames of reference of the educational process To disregard or overstress any one of them unbalances the emphasis on desirable purposes of education Accordingly, objectives may be regarded as having either a factual and impersonal aspect or a personal, psychological bearing. Thus, when we study a given domain in a purely scientific way, irrespective of the learner's personal reactions, we are mainly interested in facts, skills, organized knowledge, accurate concepts, and the like If, on the other hand, we scrutinize the way

in which the pupil behaves in a given situation, or his modes of reaction, we are led to such categories as habits of work or study, attitudes, interests, insight, modes of thinking, types of appreciation, creativeness, and the like.

A clear recognition of these two essentially different yet complementary types of objectives is one of the achievements of recent educational theory. It is generally conceded that in the past the chief emphasis was on impersonal or factual objectives. Perhaps there are extremists now leaning too far in their psychological or child-centered point of view. But educational advancement demands that due weight be given to both types of objectives

This chapter will deal with objectives mostly of the second type The discussion will be limited to those objectives to the achievement of which the study of mathematics can make a substantial contribution. No attempt has been made to list them in order of importance All are significant It is not implied that mathematics is the only study that can contribute to the attainment of these objectives, but its contributions are outstanding.

ABILITY TO THINK CLEARLY

Educational leaders have pointed out that facility with skills and acquaintance with facts do not constitute a sufficient goal for schools to set in their instruction of youth. One often hears the complaint that boys and girls are given only miscellaneous information and are drilled in routine performance when they should be "trained to think" The goal implied by the last words is very broad, and not all of the related activities are especially pertinent to the general place of mathematics in education. An attempt will be made to indicate some activities that will serve to illustrate behaviors associated with clear thinking.

Gathering and Organizing Data. Many serious problems involve the gathering and organizing of data, or should involve it if a satisfactory solution is to be obtained. In a way, this is merely a trite saying, immediately acceded to; but nevertheless

the process is not always carried out in a forthright way Data may be obscure or difficult to obtain, while in other instances personal predilections and interests invite biased selection. School experiences should impress upon pupils the fact that in many social and scientific situations an essential step in good thinking consists in obtaining the facts and organizing them Pupils should be led to realize that organization of quantitative data precedes the mathematical treatment essential for adequate understanding of the scientific principles involved While the mathematical curricula of our secondary schools have not given adequate attention to these matters, the Commission believes that in the future definite provision must be made for training of this kind.

Representing Data. In urging support for a proposition one has the problem of properly presenting the underlying data Although in the ordinary occupations of either adolescent or adult life one does not often deliberately employ scientific procedures, one should be familiar with such procedures in order to follow discussions and appraise intelligently issues that may be vital, or at least interesting, to him. Among the most promising ways of acquiring such ability is that of engaging during school years in activities demanding not only the collection but also the careful presentation of data. By such experiences the pupil should acquire the realization that the proper representation of data is often an invaluable aid to clear thinking about many problems

Drawing Conclusions. Data, well selected and properly presented, provide a basis for drawing conclusions. The difficulty of this process varies with the situation, ranging from cases in which the conclusion is simple and obvious to those in which penetration of thought or a sequence of consecutive steps of reasoning is involved Furthermore, the confidence that may be attached to conclusions varies from a moderate probability to a high degree of certainty.

For problems in many fields of man's activities, a clear and

permanent solution may be unattainable, although the answer to any particular problem may with the passage of time become clearer and more certain as pertinent information accumulates or more profound analysis is employed. Since one of the purposes of thinking is to gain conclusions, it is desirable for young people to have school experience with subjects in which the ideals of permanence and high precision are realized. This applies even for those persons whose main interest is in problems in which conclusions must involve doubt. The contrast that one is thus led to see may engender an attitude of wholesome caution. This experience may indeed leave a particularly vivid impression when opportunities are presented to study problems where a numerical measure of probability is assigned to a conclusion necessarily based on data somewhat contradictory or otherwise inadequate.

Establishing and Judging Claims of Proof. The effort to establish or judge the validity of many propositions leads to questions about the nature of proof. Any formal discussion of epistemology can hardly find place in secondary instruction, but there should be conscious experience with both inductive and deductive reasoning. The character and requisites of these two procedures should be so clearly grasped that appropriate behavior on the part of the pupil is brought about in the direction both of understanding and of making applications. In solving problems the pupil should develop the habit of asking whether he is starting from general premises and is seeking consequences, or, by examining particular instances, is aiming at universal conclusions He should seek to discover and remove ambiguity in the use of terms. He should understand the relation between assumptions and conclusions, and he should grow in the ability to judge the validity of reasoning which purports to establish proof. Proper attention must be given to generalizing these behaviors and understandings. We may then hope that pupils will apply them to situations arising in many different fields of thought

ABILITY TO USE INFORMATION, CONCEPTS, AND GENERAL PRINCIPLES

One of the chief aims of the schools has always been to impart information which can be employed in one way or another by the boy or the girl both in youth and later in adult life Certain factual knowledge is part of the essential equipment of even a moderately educated person But it is not enough, for equally important is the ability to think in terms of broad concepts and to apply general principles. Since in the secondary school a higher degree of maturity on the part of the pupil may be assumed, this school begins to differentiate itself markedly from the elementary school in the development of such capacities. Here arises one of the difficulties of the secondary school, because the varying abilities, predilections, and environments of pupils will greatly influence what the school may accomplish for them The problem is complex, but its solution must nevertheless remain one of the chief goals of education.

ABILITY TO USE FUNDAMENTAL SKILLS

To be even moderately literate there are certain skills that each individual must have at his command. He must first of all be able to read, write, and carry on certain arithmetical work. The elementary school concerns itself with such matters, but as society grows increasingly complex, technical, and scientific, the range of skills that either are necessary or are highly desirable is continually broadening. This would seem unquestionably true in the fields of language training and mathematical skills. Thus one of the essential functions of the secondary school is to foster the retention and further development of basic skills.[1]

[1] The fact that some subjects, notably mathematics, are necessary in order successfully to carry on other subjects, has led some educators to describe them as "tool subjects." This has been unfortunate, for by singling out one quality of a subject and using it as a general designation one tends to forget other qualities that are also important, particularly for certain individuals When we consider the development of civilization we see that a tendency to grow away from mere

DESIRABLE ATTITUDES

Discussion of attitudes has come into prominence in late years in educational literature. This has involved the explicit inclusion in the program of the schools of certain objectives, which, while difficult to obtain, are increasingly being regarded as among the most important educational goals. Although the term "attitude" is relatively new as a specific designation, schools have always had ideals and purposes which could have appropriately been classified under such a name. Whereas formerly these aims were regarded as by-products, to be realized automatically, the significant feature today is the attempt to analyze what is meant by these attitudes and to develop instructional procedure that will achieve what is desired.

The classification of attitudes becomes complex when carried out in great detail. One finds himself involved in an intricate mesh of ideas that are overlapping as well as words that are ambiguous. In the present connection it is not necessary to attempt an exhaustive study, so the discussion will be limited to a few outstanding aims of education that fall within the wide category of attitudes

Respect for Knowledge. The fund of knowledge now available is so extensive and varied that any one individual can become acquainted with only a minute portion of it. At the same time modern problems are so complex that most people

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tool considerations is one of its chief characteristics On every hand one witnesses the desire to transform what was originally a mere tool so that it will yield satisfactions other than utility Consequently it is quite unjustified to call studies "tool subjects" just because they have great tool values. The physicist needs mathematics, but he gains a great deal if he sees in the mathematics that he uses as a powerful implement some of the things the mathematician sees, quite as the person who is forced to use foreign language finds new satisfaction if he can develop a little of the spirit of the real language student Something is sure to be lost when anything is viewed as a tool which has in fact been merely one form in which human thought and aspirations have developed and ideals found harmonious and complete expression Such a distortion, usually arising from the desire for haste and the wish to move directly toward some objective, is likely to lead to defeat in the end.

must rely largely upon the efforts of experts to clarify them and to indicate possible solutions This means that the schools must seek to inculcate a strong and abiding respect for knowledge in order that the accumulated wisdom of the race may be preserved and effectively used Persons who have developed this attitude exhibit behaviors such as the following. They strive to make their own knowledge about problems with which they are concerned as complete and as accurate as possible They seek to settle questions on the basis of evidence rather than by personal opinion or caprice. They recognize that the judgment or opinion of those who have devoted long study to certain questions is, in general, more dependable than that of others If they cannot personally attain such standards, they are willing to accept the recommendations of experts as a basis for guidance in the most desirable courses of action Schools can help pupils develop this attitude in various ways They can assist them to distinguish between what is worth-while knowledge and what is not and can give them criteria of judgment concerning the relative merits of various types of authority. They can impress upon them the importance of ascertaining who are the authorities in various fields and the importance of turning to them for guidance. In these and other ways, the schools can foster an attitude which will make pupils eager to increase their own knowledge and sympathetic with efforts to preserve and extend the knowledge of the race Interest in a subject can result from study of it as well as motivate it

Respect for Good Workmanship Society needs good workmen, and one cannot meet this demand unless he has come to realize what good workmanship is. To some this realization comes naturally, while in others it may be a trait hard to establish. Much of the work of the modern world involves precision and minute attention to details, such qualities becoming more pronounced as civilization grows more complex Those at every level of society should be helped to attain an attitude of respect for good workmanship, and be made to realize that it is effective

and thorough work that gives one a claim to an adequate return for labor

The ability to do work well and an appreciation of the meaning of excellence are not sufficient. There must be a mental and moral fortitude to stick to a task, even when other activities seem more inviting. Men and women engaged in professions which on the whole are congenial to them spend many hours at what is wearisome and devoid of much inspiration, because an interest and absorption in what is being done often makes them oblivious of any distasteful elements. It is even a delusion of young people to believe that their "chosen occupation" will be full of thrills and devoid of all the elements of drudgery which they find in other work they are asked to do. In broad perspective, indeed, all work is likely to involve elements of drudgery Many fail to accomplish meritorious work within their possibilities solely because they do not have the quality commonly described as "stick-to-it-iveness." The majority must work for other people, and any reasonable scheme of social organization should encourage the development of efficient and dependable workers Many students of our social structure cannot imagine an organization, even though it be far more equitable and charitable than that which we now have, which will result in making people feel constantly like singing at their work Voluntary acceptance of discipline is needed in a democracy, and it is a duty of the schools to develop it as a product of the life of the school as a whole.

Respect for Understanding. Related to the idea of doing work in a creditable manner is the ideal of realizing the difference between mastery and superficial understanding. A large part of the progress of the race has been due to men who have seen in apparently familiar situations more than was evident to others What is implied by really understanding even a comparatively simple thing may be quite an involved problem. Often it includes far more than one is at first inclined to believe, for it is possible for a person to believe he understands when

in fact he does not understand at all. Modern life demands understanding as well as excellence in work or craftsmanship Education therefore has the obligation of bringing pupils into experiences where they may at least be awakened to the distinction between thorough understanding and routine performance.

Social-mindedness Here it will merely be noted that it is universally conceded that schools should provide opportunities for boys and girls to secure a wide range of social knowledge, and at the same time strive to awaken a sense of social responsibility which will remain a dominating influence throughout their lives. Unless this is done there is danger that individuals, educated within the schools that society has created and supports, will use their knowledge and capacities purely selfishly or in definitely harmful ways Thus, no matter how thorough the training that schools may give for the purpose of making people think clearly, they may be making the clever person more adroit in achieving ends either individually selfish or beneficial solely to a small class or group. Fear of such danger has led some educators to the position of subordinating the individual interests of the pupil to a purely social end. When such a position is carried too far, a personification of society may result that is almost devoid of meaning, for the implication is likely to creep in that the views of society may be ascertained as well as those of an individual. The imperative need for a large amount of well-organized instruction and training having definite social implication is generally recognized The Commission believes, however, that some of the desirable qualities of social-mindedness may be effectively aroused and stimulated by a variety of studies which in the past have not been associated with this type of emphasis.

Open-mindedness. The school has done little to remove the deep and widespread trait of being biased by one's interests. In actual fact individuals carry on reflective thinking over a wide range of activities, in many of which it is possible for them to hold a detached and scientific attitude, as they search honestly

for the answer that the data indicate. The schools must therefore aim to develop persons capable of unbiased and logical thinking and at the same time to mold character that will lessen the danger of unsocial employment of the power thus stimulated and strengthened

INTERESTS AND APPRECIATIONS

Pupils in secondary schools frequently have interests and appreciations which the school should help them develop But in addition to encouraging such personal interests, the school should seek to arouse new ones. There are at least two important reasons for this educational obligation. In the first place, the successful pursuit of any chosen activity usually requires competence in fields whose relation to his major interest the pupil has not previously perceived. Definitely related subjects should be revealed as vital to his success. He should acquire the realization that it is as difficult to isolate one activity from all others as it is for a person to live uninfluenced by the welfare or doings of other people. In the second place, interests actually outside the range of one's main endeavor may have a large influence on the satisfaction found in life Although it is to be hoped that one will find some enjoyment, or at least contentment, in the part of the daily work of the world that he carries on, one is in a sense unfortunate if, in an age so rich as the present, he does not have some active interests and appreciations Even those whose work is of an interesting and satisfying character often have absorbing avocations in which they may achieve some distinction It is still more important when work is monotonous and lacks characteristics that are inspirational to have something to which to turn for intellectual and emotional stimulus. We cannot depend upon private enterprise to furnish adequate opportunities for wholesome entertainment. Thus there is an increasing provision for recreational facilities at public expense, the public schools often aiding in such programs through their buildings, equipment, and personnel The

fostering of interests and appreciations in pupils while in school, which is being stressed here, has a somewhat different purpose, for it aims to make people, in some ways, at least, more self-contained, more resourceful, and less dependent upon private or public mass entertainment facilities as the sole means of relaxation and enjoyment.

At first sight it might seem that the fostering of purely personal interests and appreciations is giving education an individualistic bias at a time when educators are stressing the importance of social attitudes. It would seem to run counter to the contention that everything a boy or girl does in school and everything studied must have a clear and unmistakable value for society On the contrary, however, the cultivation of personal interests need not be anti-social. The individual who finds interest in his cultural heritage and is well acquainted with it, is very likely to see the present more clearly than one who believes that the problems of today can be detached from the past The Commission believes that social attitudes are likely to manifest themselves in people who have an interest in such domains as literature, art, music, science, and mathematics. These have been among the truly humanizing activities in which man has engaged. They reach his finer rather than his grosser nature, and through their influence a nobler and more kindly society can be built

In a democratic society it is imperative that there should be widespread appreciation and knowledge of subjects other than those that have to do with political or social matters, or are requisite for professional or vocational competency. There should be strong bonds between groups working at diverse occupations, for in no other way can we be certain of retaining our cultural heritage, to say nothing of increasing it. There is no doubt that art, literature, and science will flourish in a state where all possess sufficient knowledge of these things to have an appreciation and a liking for them; but there may well be doubt whether the necessary public support will be given if the schools try to train

only specialists and pay no attention to thoroughgoing general instruction. Through their high school studies, boys and girls of ability can be made to feel that they are partakers of the significant achievements and experiences of the race, their common heritage bringing them together and mitigating the disjunctive influences of varied avocations and special interests.

One should distinguish between those types of *detailed knowledge* which may disappear quite quickly after the study of the subject has ceased and the general *appreciation* which remains. A pupil who studies chemistry for only a year soon forgets the formulas and most of the facts he has learned; within a short time he could not pass an examination and he would be useless in a laboratory unless he devoted some time to review. But chemistry will not be throughout his life a mystery, a mere name, or at most a vague dictionary definition. Having been an active participant in chemical study, and having himself performed chemical experiments, he knows in an intelligent way why chemistry is of great importance to human welfare, and he is familiar with the basic theories of the constitution of matter and with methods of investigation by which chemistry reveals many of the secrets of nature. To him the chemists of the world are not like workers in a land he has never visited and of which he has no conception whatever He might truthfully say in after years that he has never used in any vocational way the chemistry he studied, just as he may say he has never used the pictures on the walls of his home in any such sense But if he says with calm and deliberate seriousness that the study of chemistry meant nothing to him, he is merely making a highly damaging admission and is not in fact criticizing chemistry. A similar statement can be made concerning the study of any other great field of learning For example, a pupil should derive from the study of trigonometry an appreciation of the importance of the subject in physics, engineering, surveying, and astronomy that will remain after his own ability to solve problems has gone. The great potential values of the residuals that remain after detailed

knowledge is forgotten has been well stated by Professor Snedden.

> The purpose of the teaching given us in these fields was surely never designed to make astronomers, African explorers, writers of poetry, or painters of pictures out of us But we are cultured persons to the extent that we have rich appreciations precipitated from such long range contacts as we were able to make in these great human enterprises [2]

Studies which give a general cultural background can also create interests to be carried on later as leisure-time activities Although much is heard about the necessity of educating for profitable use of leisure, the aims are often too low or too circumscribed, as Professor Briggs has observed:

> Unfortunately, the prevailing conception, even among educators, of leisure time activities, is that they are primarily hiking, games of various kinds, creative work that is more or less artistic, and improved association with one's fellows.[3]

In short, the aims are often such as can be attained effectively without any considerable school experience, and hence are not ministered to by any supposed "educating for leisure"

OTHER OBJECTIVES

Other objectives are frequently mentioned, of which health is a prominent one There can be no question of its importance, and it is passed over here solely because the subject with which this report deals is not especially concerned with it. One remark, however, will be made There is an increasing amount of knowledge on the part of the public as to sanitation and hygiene, and people gain information concerning medicine or health through agencies other than the schools, as is proved by the rapidity with which they have become vitamin conscious. Thus in the matter of health instruction, the responsibility of the

[2] Snedden, D S *Educational Sociology for Beginners*, p 520. The Macmillan Co, 1928

[3] Briggs, Thomas H. "The Philosophy of Secondary Education" *Teachers College Record*, Vol. XXXVI, pp 593 603, 1935

school varies greatly from child to child, and in the case of children from some homes might even be completely dismissed.

Much has been said concerning citizenship and worthy home membership as objectives. Such broad terms must be analyzed before they have meaning. Citizenship should certainly mean more than an alertness to current problems, an awareness of civic responsibility, and a willingness to aid in worthy enterprises. It should embrace cultural interests and appreciations, and should imply a high degree of competence in the actual work in which one is engaged. One may recall profitably that Benjamin Franklin, often described as the world's greatest citizen, was an eminent philosopher and scientist as well as a faithful servant of the people. No small amount of the influence which he was able to exert, and the confidence he could inspire, came from the esteem in which he was held because of his brilliant accomplishments in a number of fields.

Worthy home membership should likewise imply knowledge, abilities, and appreciations quite as well as the long cherished domestic virtues, if the word "worthy" is to mean much. For instance, elementary mathematical knowledge can be regarded as essential in the home, and mathematics more advanced can, on occasion, be desirable. The objectives previously discussed can all, in fact, contribute to worthy home membership as well as to citizenship, if these latter objectives are raised to a desirably high level.

CHAPTER III

THE PLACE OF MATHEMATICS IN EDUCATION

"I shall on with my story of praise, and then show you the heart of my message."

—VIOLA IN TWELFTH NIGHT

THE influence that mathematics has long had in our civilization and its growing importance indicate in a general way the place it should occupy in education. Since the function of the schools is to equip boys and girls not only to be effective members of our society but also to be appreciative of our culture, schools must especially provide contacts with a study which has done so much toward both "controlling our environment" and forming our intellectual background. Though such a general statement is easy to make, difficulties arise when its consequences are sought in such specific things as curricula For illustration one needs merely to call attention to the divergent views now expressed. On the one hand there are those who urge that only a small amount of mathematics be universally required, and who say that we should expect only pupils with special inclinations to go further. On the other hand we have the thesis of Hogben, not a teacher of mathematics but a social biologist, that there are urgent social and individual reasons for a large number of persons to become proficient over a wider range of mathematics than they have covered in the past.

One point should be disposed of at the beginning. There are many persons occupying important places in society who to all appearances have negligible mathematical appreciations yet live rich cultural lives. In some cases their study of mathematics evokes unpleasant memories, tempered only by recollection of

the joy that accompanied ultimate release. What does the Commission make of this? In many cases the dislike for mathematics may have been created either by the ineffectiveness or the personality of a teacher or by the unsuitability of the material that a competent teacher had been required to present. This, however, cannot be offered as a universal explanation. If mathematics is to be given the prominence in education which this Commission believes should be given it, every effort should naturally be made to reduce the number of those who carry on the subject in secondary years with a feeling of unhappiness and with a belief that no substantial benefit is being obtained. Better courses of study and better teachers can do much to bring this about. It can be said here, however, that the Commission is prepared to accept the possibility of conditioned antipathy in some individuals, and would not force mathematics beyond the elements of arithmetic either upon a pupil whose rebellious distaste toward the subject *seems firmly* entrenched or upon one whose genius or legitimate absorptions leave little leisure for mathematical development.

As to the complaint that there are too many cases where mathematics has been unsuccessfully studied, the following needs also to be said. Similar criticism is made of the teaching of other subjects, so the complaint is only a part of the popular criticism of current efforts at educating. We "teach" English, and there is still much bad grammar and an apparently increased reading of cheap and vulgar writing. We dwell upon the social studies, and their lessons are left within the classroom by many who succumb to the lure of economic panaceas. We instruct in health, but the rules are disobeyed not infrequently by the teachers themselves as well as by the physicians who devise them. These are discouraging facts about human nature, but they are not reasons for lowering our educational standards. The Commission believes strongly that educators should not resign themselves to the doctrine of "minimum education" as the norm. It believes that we should by all means require as ideals and stand-

ards something definitely superior to the small amounts of this or that subject which some people "get along with." Constant reasonableness should be used in meeting the difficult special situations that grow out of mass education; but the schools should certainly be unremitting in their efforts to raise the general standard of American culture. We should never be content with a "high standard of living" only in the material sense

There is a positive answer to the question as well as the somewhat negative answer just set forth. The Commission believes that under proper teaching, supported by discriminating and sympathetic guidance, a fairly large proportion of boys and girls can realize that man has lived so long, accomplished so much, and learned so many things, that they cannot reasonably isolate themselves from traditions which strongly condition the present. The conviction will come to them that they can hope to deal effectively with the future only by paying attention to the past. In view, therefore, of the increasing importance of mathematics to civilization, because of the techniques it has perfected as well as its methods of reasoning, the Commission believes that ample opportunity and encouragement should be given to all individuals to continue their mathematical training as far as their powers allow and as other conditions permit. Such instruction should be in definitely organized mathematics courses, for incidental learning of mathematical fragments in connection with other studies cannot give either the general understanding or the appreciation of the subject that is here advocated.

The manner in which mathematics as a school subject contributes to the objectives discussed in Chapter II will now be discussed.

MATHEMATICAL STUDY AS TRAINING IN CLEAR THINKING

It is unfortunate that for a long time it was maintained that mathematics furnished a general training of the "reasoning faculties," as if a certain power might thereby be developed to

function in all situations. The problem of "transfer," however, need not be gone into here more than to note that it is now generally accepted that transfer is possible. It is to be observed, moreover, that nothing was said regarding the training of a general "reasoning faculty" when in Chapter II the ability to think clearly was discussed.

What was said about *gathering and organizing data, presenting data,* and *drawing conclusions* shows at once how important mathematics may be in giving instructive experience in these procedures. If, as was stated before, mathematical teaching in the past has not paid sufficient attention to the first point, gathering and organizing data, this was partly due to the fact that scholars in other fields were not always cooperative. They shut themselves away from mathematical methods. But this has now changed, and the development and the wide use of statistical methods have greatly increased the area in which significant quantitative work is possible. A certain knowledge of basic mathematics is required in these areas. A fair competence in algebra is needed if one is to understand concepts and procedures beyond the most elementary ones In a way algebra may be a "tool" for statistical work but a tool in a very fundamental sense, since it is woven closely into the texture of the subject and into the thinking which is involved. That something far more than routine skill is required becomes apparent when the subject is carried into the range of *probabilities,* a field into which it inevitably moves as soon as measures of reliability are introduced.

The truth of what has just been said is being constantly demonstrated by individuals attempting to do statistical work without adequate preparation. The difficulties which so frequently take them to a mathematician for aid usually center around the *meaning* of things Actual use of formulas may have caused no trouble, but neither the ideas from which the formulas were derived nor their implications are understood. The person in distress has usually obtained a number of whose accuracy he is

sure, but whose meaning quite confounds him. Such a situation is evidence not of lack of numerical adroitness but of comprehension. The person may be bewildered solely because he does not have at his command the only language which will allow one to "think through" the subject he is trying to handle. Such unhappy situations as this will be corrected only when mathematics is rightly viewed as essential to clear thinking in certain domains, and all talk of it as only a "tool subject" has ceased.

Portions of mathematics can be made especially effective in developing habits and traits discussed under the heading "Establishing and Judging Claims of Proof." Geometry has always been regarded as presenting unusually impressive instances of deductive reasoning. In a formal course in the subject a pupil almost daily has an assignment involving "proofs," and in no other study is this likely to be the case. The nature of the material with which the proofs deal is indeed quite different from the "life situations" which he will encounter later as an adult and a citizen. It is altogether probable that, in the past, mathematics teachers did not do all they should to make the experiences of the geometry class as broadly significant as they may well be.

Geometry has been treated solely as geometry and not as a subject, which in addition to being a splendid example of deductive reasoning, important and interesting in itself, can also serve the purpose of creating a critical attitude of mind toward deduction and thinking in general. It is essential to have the theorems of the text understood and the problems worked, and to place the main emphasis of the study upon geometry itself; but it is important also for mathematics teachers to make geometry yield all the educational benefits it can. Teachers who have experimented have found that, without lessening seriously the amount of geometry taught, the course can be made the means of establishing a general critical attitude on the part of pupils, an attitude that they recognize and value. Especially at the high school level, principles of deductive thinking can be most effectively

taught in connection with a well-organized, substantial subject such as geometry, which, being logical itself and free from personal prejudice, can serve as a yardstick. It is clear that little can be accomplished merely by announcing principles and criteria of good thinking and calling attention to the danger of their violation by illustrations of good and bad thinking taken from "life situations." In a miscellany of such illustrations there would be neither coherence, growth, nor any body of knowledge significant in itself Furthermore, abstract principles of reasoning are not designed to arouse response, particularly in young people. On the other hand it is not strange, when one pauses to reflect about it, that geometry, with its origin in mensuration constantly kept before us by its very name, with its employment of figures and its superb logical structure, often has been a favorite study and has even stirred those gifted with literary expression, though not pursuers of mathematics, to affectionate praise of its satisfying truth and its serene beauty.

Until recently there has been little inductive thinking in elementary mathematics Comprehensive books on algebra have frequently contained a chapter with the austere title "mathematical induction," probably poorly understood and productive of little result unless the pupil went considerably beyond algebra Mathematicians have disagreed with the statement of Huxley that mathematics knows nothing of observation, of experimentation, or of induction. Though definitely untrue, if one is thinking of the way in which the subject has developed, the criticism has been valid when applied to methods employed in its teaching There is now, however, a definite trend toward leading pupils into new topics through their own experiences. Especially is such procedure possible in geometry, and the appearance of informal geometry, including intuitive and experimental procedures, in the seventh and eighth grades is a distinct and significant step in this direction The possibilities are certainly numerous, and it is to be hoped that mathematics will emerge finally as the vehicle through which may be obtained

impressive experiences in inductive as well as deductive reasoning.

MATHEMATICAL INFORMATION, CONCEPTS, AND PRINCIPLES

A large body of mathematical knowledge is of unquestioned utility and involves no intricate techniques. For example, theorems of geometry are merely facts about figures, of either a descriptive or a metric character. Necessary for certain trades or technical occupations, they may be universally desirable Undoubtedly a very large number of Americans can find the area of a rectangle. To find the area of a circle, however, would lead many to ask aid, although some would be quick to claim, "I could solve such a problem once" However, just as one need not inventory the mathematical requirements of trades and technologies, he need not enumerate all instances in which mathematical information may profitably be used, in order that the place the subject deserves in our school curricula, on the score of information alone, may be recognized.

On a somewhat different level, insofar as they affect mental activities of educated persons, come the concepts and principles with which one has contact in mathematical instruction When well impressed upon the mind, they are more permanent possessions than facts or even skills, which are likely to be impaired through disuse In a society which draws so heavily upon mathematics, as does our own, mathematical principles and concepts should affect the manner in which the individual thinks and should color the appraisals he makes. The signed numbers of algebra, the similarity, the congruence, and the parallels of geometry, the rates of change of elementary calculus should be so taught that they leave lasting impressions. The ambition to make mathematical instruction more broadly significant through emphasis on concepts has led to stressing the *function concept* as a unifying element Inasmuch as it deals with relationships, it is quite true that few concepts have greater univer-

sality or importance. A society, all members of which while in school have been given persistent and effective contact with this concept, should view problems and situations more intelligently than a society which has only a certain number of mathematical specialists. But the great importance of the function idea should not lead to an over-emphasis upon its significance, nor should it lead to slighting mathematics which does not come under its scope, for very important and very interesting parts of the subject are unrelated to it.

MATHEMATICAL SKILLS

An effort has been made to describe the fundamental ways in which mathematics is embedded in thinking and to discuss its concepts in such a way that they will not be confused with skills. But a precise dividing line cannot be drawn. Mathematical technique is a very real thing. Its extensiveness and the difficulty of mastering it will always be a source of discouragement to some pupils and a perplexity to some teachers. Unless one has facility with its processes, however, mathematics cannot be used effectively. Its techniques must be so well acquired that, in a sense, they take care of themselves, leaving all of one's powers available for other purposes, especially the higher ones of analyzing and directing. If the handling of fractions or the solution of simple equations taxes the pupil's ability, there is little chance that he will deal satisfactorily with situations in which these processes enter. If there is, on the other hand, such command of the processes that there is true fluency in their use, there is likelihood that mathematics may be justly appraised and effectively used.

The character of the techniques of mathematics is in part responsible for the fact that it is necessary to study the subject for some time before it pays extensive returns. A pupil may attend one class in first aid, learn how to apply a tourniquet, and later save a life But it is hard to conceive of much benefit from one lesson in algebra or geometry. The deep-seated desire for quick

returns, however, will always arise in connection with mathematical study as in other situations, and should be honestly faced Of course every effort should be made to make mathematics pay returns as promptly as possible, even the very prudent investor with his eye chiefly on the future is pleased with early dividends, for they support his faith in large ultimate profit. The recent efforts to make mathematics, especially algebra, yield more of interest in its early stages are both laudable and notable. They make the study resemble an insurance policy with a good surrender value in early years.

Though techniques should be regarded as means and not as ends, those of mathematics have certain virtues on account of the broad educational processes involved. Some teachers, unsettled by attacks upon mathematics, have sought to turn these attacks aside by fleeing from techniques as though they were evil, or have pretended that they could be learned, to the extent necessary, without conscious or serious effort. The actual educational value of the techniques, which makes such a retreat unnecessary, will be touched upon later.

It is not difficult to draw up a list of situations in which one can profitably use algebra, and such a list assists in a realization of its importance. One is confronted, however, with the question whether the mathematics that could be used actually will be used even when the person is competent. Certainly not always by any one person, even the accomplished mathematician may not employ his knowledge in all situations where he could use mathematics. It is to be expected that in actual life mathematics will be used according to the individual's taste and the extent to which he is actually stimulated by some problem. Neither mathematics nor any other subject can make the horse drink. The mathematics teacher is as powerless to make a pupil use mathematics as is the teacher of health to make him follow its well-established laws. But education, which must not shirk the responsibility of developing capacities, at the same time must enable persons to realize the meaning of intelli-

gent living in a scientific age and urge them not to slump into a state of mental indolence, after having been potentially brought to a rather high level of understanding.

Comparatively few pupils during the secondary years know what their later activities and studies will be. In the absence of required courses or suitable advice, many erroneously assume that they will need no extensive work in secondary mathematics. On entering college they often find doors closed to desired fields of study because of the lack of adequate mathematical preparation For example, they are unable to take work of substantial character in the physical sciences, or in those parts of the social and biological sciences that employ statistical methods. To be sure, special courses are sometimes offered for poorly prepared students, but such weak or emasculated studies are poor substitutes for standard courses that employ mathematics where it is naturally needed. To postpone until college years basic preparatory studies that experience has convincingly shown can profitably be pursued in the secondary school, gravely handicaps the pupil in his later effort to make a program of real collegiate studies. The doctrine of "postponement," like the doctrine of "incidental learning," however alluring to the shortsighted person and however valid in certain subjects, is indefensible in the case of mathematics. The subject is so extensive and so difficult, requiring systematic and protracted study, as to be unsuitable for the general application of either of these doctrines.

Administrators should feel deep concern over the large number of pupils who find themselves without adequate preparation for the activities, professions, or additional studies which they later wish to undertake but which presuppose substantial foundations In their solution of this problem lies the test of their educational statesmanship. A strong and corrective influence should be exerted upon those boys and girls who are capable of doing fair work with secondary mathematics but who, although they have no serious dislike for the subject, and insufficient knowledge to form a sound opinion, think it unnecesary and

yield to what is easiest or most glamorous at the moment. It is for the good of society that each year there should go forth from the secondary schools a large number of young people with marked proficiency in the technical skills of mathematics, together with a fundamental understanding of some of its concepts. The steady flow of such a group into our democracy is a major responsibility of school administrators.

MATHEMATICS AND DESIRABLE ATTITUDES

It is often asserted that as a people we have high regard for the specialist, the claim seeming to indicate a respect for knowledge and competence. However prevalent such a regard may be, it is not a discriminating one. Our people are, in fact, frequently and badly victimized by pseudo-specialists and pseudo-experts, simply because they are unable to recognize important fundamentals. In the future years, as in the past, revolutionary programs of an economic and social nature will be urged upon our country After the partisan and political character of a proposal has been scrutinized, there will still remain the question whether the proposal is based on a broad knowledge of facts and an intelligent analysis of their relationship, or on little more than wishful thinking.

Now no subject excels mathematics and those sciences that draw heavily upon it for ability to set up high standards of knowledge, of analysis, and of techniques for arriving at accurate conclusions. Since mathematics and kindred sciences undertake to construct systematic bodies of doctrine, they stimulate thinking, analysis, discovery, and growth. When the pupil is studying them, he finds himself unable to advance merely by using his memory or his fluency of speech. He uses books whose first chapters *must be understood* in order to advance to the last. Only by comprehending each step in the progressive development of these sciences can he finally attain the understanding of the things that are being studied, things that are permanent and significant From this contact with ideal knowledge, he gains an

experience that furnishes a background for accurate discriminations and distinctions. We are at present confronted by so many social problems for which it is impossible to "know the answer" that it is especially important for society to acquire the steadying and careful habit of procedure which may come through the discipline of mathematical study. Good will and a warm heart are not enough to furnish us with the protection of life insurance; the formulas and the tables of the actuary are in some ways more necessary. This Commission believes, in short, that mathematics can be so taught that it will help reveal the meaning of knowledge, as distinguished, on the one hand, from opinion and conjectures, and, on the other hand, from trivial and commonplace facts.

It is probably true in certain respects that pupils are held to a higher level of achievement in mathematics classes than elsewhere by the nature of the subject itself, for here the standard of accurate definitions, of logical coherent statements in demonstration, and of precise results must necessarily be stressed. In a broad sense, all this is merely claiming that mathematical methods furnish examples of good workmanship. The possibilities of giving discipline of universal significance are not lessened by the fact that not all the situations in which it may be used have the general characteristics of mathematics The question is one of creating an ideal and of setting up standards of excellence, not one of whether mathematics resembles other subjects. First the ideal must be glimpsed, then it must be so lived with that it will become a part of our lives This practice can most likely be accomplished through formal educational experience in which the ideal is constantly emphasized. This Commission believes that some teachers of mathematics have been indifferent to their full opportunity to hold up to their pupils high ideals of workmanship College students, showing very clearly that they have been allowed to "get by" in mathematical work as well as elsewhere, will unblushingly hand in papers whose careless appearance should give shame to any student Neat, well-arranged

work can have a sensory appeal, and should stimulate accuracy and precision. All these qualities, neatness, accuracy, and precision, are merely attributes of "taking pains," a thing that is essential to good work in any field. It is not to be expected that mathematics will maintain the place its teachers desire, unless teachers are willing to meet the insistence of administrators and educators that their subject yield all possible contributions to the varied goals of education.

Intimately related to good workmanship is the ideal of thorough understanding, and certainly here even the techniques of mathematics can be made to contribute wholesome lessons. Something as non-essential as a change of letters will often cause confusion for the beginner. Though a pupil may think he understands the identity $(a + b)(a - b) = a^2 - b^2$, he is not likely to argue the case if he did not see its application to 84×76 until it was pointed out to him. A pupil's progress in mathematics depends largely upon his grasp of the full implication of both the concepts and the powerful techniques. When he fails to understand some point, his teacher frequently can trace the difficulty back step by step to something the pupil believed he understood but which in reality he grasped only imperfectly. The lesson to be learned from such an experience is much broader than the mathematics involved. Many parts of the subject can be so presented that mathematics will help to build up in pupils the desirable habit of questioning the clarity and the fullness of all their knowledge. The desire to look into things a little more deeply can be stimulated by frequent illustrations in mathematics of the profitableness of such a search.

It might seem at a first glance that mathematics has little to do with developing social-mindedness, which was set down in Chapter II as one of the attitudes much in favor at present. The "social studies" are currently urged as the cure for maladies which we know afflict us, but which we cannot describe except in vague terms. It is, however, to be remembered that mathematics sprang from elementary human needs, from problems of

feeding, clothing, and shelter; and today, although it embraces a great deal more, it still has intimate connections with such primary questions. Thus, when properly taught, mathematics surely should appear as one of the chief instruments of "social progress."

Open-mindedness, the last attitude discussed in Chapter II, is related to willingness to admit one's self wrong, but it is an attitude that should be carefully distinguished from mere mental instability. In mathematics a person cannot long deceive either himself or another, for arguments that arise must end by the admission of one disputant that he had been wrong. This may cause the discomfort that some pupils experience in the mathematics classroom, where it is difficult to cover up a weakness Even if one's error is honestly come by at the expense of considerable toil, it must be laid aside without prejudice or resentment. Every pupil in geometry has had the experience of seeing what he believed was a valid proof of an "original" explode under the questioning of his teacher. Even in studying his lessons, he repeatedly finds it necessary to abandon a thought that for a few moments held the prospect of being the key to working a problem or proving a proposition. Such a situation may look somewhat grim, but, though the discipline may be too unpleasant for a few, an actual laboratory in open-mindedness, where one receives training in admitting an error endeared to him by the effort expended in seeking to establish it, may be more valuable in forming a social attitude than rhetorical exhortations

MATHEMATICAL APPRECIATIONS

The schools must teach mathematics beyond its elements not only to equip those who need it as a tool, but also to make people appreciate in a forthright and intelligent way how basic is its place in our culture. The artist who depicted the "tree of knowledge" for the Hall of Science at Chicago placed mathematics at the very roots, with other subjects growing and flowering from it, thus implanting in the minds of many people the

idea of its important cultural value. But such a mode of instruction is not sufficient, nor is it sufficient for teachers of mathematics, animated by enthusiasm and pride, to display their trees of knowledge in their classrooms. The goal should ever be to create such understandings and appreciations that pupils in their own right become competent appraisers of mathematics The position of mathematics in the "tree of knowledge" is a challenge to teachers of mathematics as well as an assertion of the value of their subject, and, repeated in our classrooms, it is a challenge to pupils, showing them not only what they may need in order to succeed, but what they must know if they are to comprehend certain essential elements of the civilization they are to share. The "tree of knowledge" is suggestive of a library with the names of poets, dramatists, essayists, and novelists cut into its walls Assuredly the highest purpose here is not to give merited honor to those who cannot profit from it, nor to elevate ourselves a little by the act of honoring them, but rather to encourage those who see the names to learn why these names merit the honor shown them, and to share in a feeling of appropriateness

In the past mathematics has enjoyed such an appreciation, shown by the honors paid it in our literature and philosophy, and it still finds honor in contemporary literature. The fact that there are people who study mathematics and then assert that their time was wasted does not affect the matter at all. The cultural tastes, appreciations, and accomplishments of the critic should be appraised before weight is given his opinion, though naturally in practice charity frequently intervenes. The Commission has stated at the beginning of this chapter that only the bare elements of mathematics need be required of a pupil thoroughly rebellious against the subject or of a pupil with a strong talent, wishing to devote his time to other subjects. The Commission believes, however, that appreciation of mathematics should be even more extensive than it now is, and it strongly urges upon administrators the wisdom and importance of using

as a general guide the ideas so well expressed by Professor Snedden in the lines quoted in Chapter II [1]

There is evidence of some interest in mathematics among adults for reasons other than narrow utility; for example, news papers and magazines from time to time carry mathematical problems Although these problems sometimes have the nature of a puzzle, they reveal an inclination toward mathematical thinking One would hardly care to venture the prediction that mathematics will ever be a serious leisure-time activity for adults to any extensive degree, but we have no knowledge of what the actual possibilities are Encouraging success has at tended some notable efforts to popularize the subject, and a thorough consideration of the problem of adult education may well show that the mathematics taught in the secondary school is necessary for desirable work in later years.

REMARKS

The Commission believes that what has been said indicates that mathematics should have a prominent place in secondary education There should be ample provision for courses beyond the ones that are required, conscientious efforts being made to influence pupils to continue mathematical study. Boys and girls should be informed as to the number of subjects that employ mathematics, and they should be led to see that an acquaintance with it helps one to live more intelligently in an age as scientific and as technical as our own. They should also be informed that in addition to its great helpfulness to man, mathematics has peculiar qualities, representing a form of perfection so striking that many persons, although not following it in any professional way, have considered their study of it one of their most valued experiences Nor should there be withheld from them the noteworthy fact that the greatest tributes to the subject have come not from mathematicians but from dramatists, poets, and novelists Finally, they should be protected against

[1] Page 33

becoming victims of the doctrine of incidental learning or the doctrine of postponement

The effort to make mathematics a prominent feature of education implies the desire to keep secondary education on a high level, with respect to both the ideals that inspire it and the standards of achievement that are expected The Commission supports such a general educational position, in opposition to the movement toward minimum education. It is true that our attempts at universal education will bring into the schools many a case of Johnnie Lowique and Winnie Barely-pass, as well as delightful Huck Finns who "take no stock in mathematics." Unless the American temper changes completely there is little danger that such boys and girls will be dealt with unsympathetically while we continue our efforts to devise studies and activities really suitable for them. But they should not be allowed to set the general pattern for education, any more than their tastes should be allowed too much weight in determining the pleasures and diversions available for educated people. A survey of movies and radio might indicate, however, that such a surrender is being made. In concluding, one might say that the statement of Hogben that the history of mathematics is the mirror of civilization suggests that its position in the schools may reveal something not only about our conception of education but about our national philosophy and ideals.

CHAPTER IV

THE MATHEMATICS CURRICULUM

"We shall make one or two postulates, deduce rules, and give examples."

—FOWLER, THE KING'S ENGLISH

BASIC CONSIDERATIONS

The Importance of Continuity and Organic Growth. For many years American educators have urged the creation of curricula based on the view that the formal educational process extends from the pupil's infancy through his adolescent years, or even further This idea is now receiving uniform emphasis in all the major subject matter fields Mathematics, by virtue of its highly cumulative nature, cannot yield the cultural returns of which it is capable if it is offered as a disconnected sequence of isolated units. On the contrary, the major objectives of mathematics, its central themes and its broad life values, must be given an opportunity to unfold gradually and continuously, in harmony with known facts of mental growth.

The Importance of Flexibility Secondary education in America has not yet achieved relatively stable, clearly defined lines of demarcation that set it off from the domain of elementary education on the one hand and from that of collegiate education on the other [1] We have, and may continue to have, a diversity of administrative divisions that are due to a variety of considerations Thus, in certain communities, the traditional 8–4 plan has been replaced by the 6–3–3 plan, the 7–5 plan, the 6–6 plan, or the 6–2–4 plan, either specifically for curricular

[1] In this report the junior college is regarded as belonging to the secondary field See page 150

purposes or because of a problem of building accommodations.[2]

At this point we are merely interested in the fact that because of this variety of types of administrative organization in the field of secondary education it has been extremely difficult, even if it were desirable, to arrive at anything approximating general agreement as to curricular offerings at any stage of the educational process. No national commission can prescribe, or should even attempt to suggest, such a rigid organization of the materials of instruction that there would be no further opportunity for individual initiative and constructive experimentation. On the other hand, there is need at the moment to elaborate as carefully as possible a group of guiding principles that may offer a definite step toward educational harmony A tentative list of such principles will be outlined in the following pages. The second step is that of describing the curricular offerings, as far as possible, in terms of broad fields rather than in terms of specific units of work

It is the opinion of this Commission that the obvious difficulty of providing for both continuity and flexibility has been the great stumbling block in the development of a nation-wide mathematical program of instruction Accordingly, in this Report is described a program for mathematics in grades 7 to 14 that definitely aims to provide for continuity of development, and that at the same time respects the reasonable demands for flexibility on the part of school administrators and teachers.

The Work of the Elementary Schools. The Commission has not attempted to suggest mathematics curricula for the first six grades. That responsibility has been assumed by the National Council Committee on Arithmetic,[3] which will issue a separate report dealing with this problem.

[2] Data concerning the growth of the relative frequency of these administrative types of organization may be obtained from the publications of educational research bureaus, and from educational yearbooks and statistical abstracts See, for example, the monographs comprising the *National Survey of Secondary Education*, Bulletin, 1932, No. 17, U. S Office of Education

[3] Sponsored by the National Council of Teachers of Mathematics, under the chairmanship of Professor R. L. Morton, Ohio University.

The mathematics program of the elementary schools is the indispensable foundation of all the pupil's later mathematics work. If that foundation is weak, the pupil's subsequent progress is likely to be permanently handicapped.

In the following pages, it is assumed that a pupil who is adequately prepared for the work of the seventh grade has acquired a working knowledge of the arithmetic commonly taught in the primary schools. An increasing number of representative syllabi also assign to the elementary grades some preliminary work in the field of space intuition and a knowledge of geometric forms. Hence the following attainments may be regarded as the normal mathematical equipment of the American pupil who has satisfactorily completed the work of the sixth grade:[4]

(*1*) A familiarity with the basic concepts, the processes, and the vocabulary of arithmetic.

(*2*) Understanding of the significance of the different positions that a given digit may occupy in a number, including the case of a decimal fraction.

(*3*) A mastery of the basic number combinations in addition, subtraction, multiplication, and division.

(*4*) Reasonable skill in computing with integers, common fractions, and decimal fractions.

(*5*) An acquaintance with the principal units of measurement, and their use in everyday life situations.

(*6*) The ability to solve simple problems involving computation and units of measurement.

(*7*) The ability to recognize, to name, and to sketch such common geometric figures as the rectangle, the square, the circle, the triangle, the rectangular solid, the sphere, the cylinder, and the cube

(*8*) The habit of estimating and checking results.

[4] For a synopsis of present tendencies in arithmetic, the reader may be referred to the *Tenth Yearbook* of the National Council of Teachers of Mathematics See also Norton and Norton *Foundations of Curriculum Building*, Chapter XI Ginn and Co, 1936 For the work of the primary grades, see especially, Morton, Robert Lee. *Teaching Arithmetic in the Elementary School* Silver, Burdett Co, 1937

A TENTATIVE LIST OF GUIDING PRINCIPLES

Considerations Governing the Selection of the Materials of Instruction, Grades 7-12. Among the considerations governing the selection of material in building a curriculum, the following may be mentioned

(*1*) Since there has been general agreement that the learning of mathematics rests upon acquiring a knowledge of a certain body of concepts, principles, processes, and facts that are essentially the same for all pupils, there can remain little freedom of choice as to the inclusion in the curriculum of these fundamental elements. The curriculum should include the basic elements of arithmetic, algebra, geometry, graphic representation, and trigonometry.

(*2*) In contrast with this permanent foundation, we have the equally important fact that mathematics has had a remarkable growth and has been extended to widely varied fields of application.

For every type of pupil, a mathematical course of study must give constant attention to the "foundations," while at the same time it stresses significant applications within the learner's potential range of understanding and interest

(*3*) The selection of the fundamental mathematical units of work, especially in grades 7-9, is a highly technical task.

In particular, the fundamental concepts, principles, and skills of mathematics must be introduced and developed in a carefully organized pattern Due attention must be given at all times not only to logical considerations, but also to psychological and pedagogical principles.

(*4*) Extensive experience has led to the conviction that in the case of retarded pupils, modifications are needed in the rate of progress and the degree of comprehension, rather than in the choice of the basic mathematical units. (See also Chapter VII)

(*5*) The precise scope and degree of emphasis to be given to each major type of work, in a particular school, cannot be

stated with finality in any general discussion On the contrary, these items must be regarded as subject to further local experimentation, in the light of actual time schedules and of desired or possible types of application and of training.

(*6*) Psychological considerations such as those having to do with the problem of understanding, with motivation, rates of learning, and with degrees of mastery, are also of great significance in connection with the construction of modern curricula These questions must certainly be kept in mind when one wishes to determine the amount of work which may safely be accomplished during a certain period of time They are furthermore of primary importance in the preparation of detailed instructions for the teaching of each unit or topic.

(*7*) Mathematics is often described as a "hard" subject. It has acquired this reputation (i) because it is composed of a relatively large body of closely related abstract ideas often presented too abruptly, (ii) because its fundamental facts and principles must be learned as an organized sequence; (iii) because only constant attention and real understanding will lead to success in this field; and (iv) because only considerable practice, over a period of months or years, will insure mastery and the ability to apply with ease the results of mathematical training. While these features of mathematics cannot be denied, it is also true that each forward step in the subject is, as a rule, a very simple one. Hence, by safeguarding each day's progress, and by following a teaching practice based on the laws of learning, the teacher can eliminate, to a large extent, the painful and futile struggle that is only too evident in some mathematics classrooms In particular, the following considerations are significant:

(*a*) Early in each year the mathematical maturity of each pupil should be determined In case the required information is not available from reports, inventory tests may be needed to determine the amount of ground that may be covered during the semester, as well as the necessary amount of reteaching

(*b*) Since mathematics is a cumulative subject, pupils should

be made to realize that each day's work counts toward success or failure

(*c*) An understanding of the concepts and principles of mathematics is the key to its successful study. To teach in such a way that the concepts become clear is the hardest and the most significant task confronting the teacher of mathematics. By way of illustration, a definition should usually be the outgrowth, not the beginning, of a learning process

(*d*) "Overviews" and motivating discussions are valuable as directing guides, while summaries and organic reviews are effective means of creating perspective and confidence A properly constructed curriculum will give adequate attention to such considerations

(*e*) In the past, much dependence was placed on mere drill. Recent psychological investigations suggest that all techniques should be based on insight. This implies that adequate practice is to be provided, not mere drill, to lead the pupil to proper assimilation and mastery.

(*f*) Modern psychology has proved the effectiveness of "spaced learning" That is, "bunched learning" is not so productive of lasting values as "spaced learning" With slow pupils, especially, the idea of a periodic return to the same topic, providing for its growing mastery and enlarged application, is of the utmost importance Experience shows that we cannot expect "one hundred per cent mastery" after a single, brief exposure.

(g) The slow learner profits by at least the same degree of motivation, of cultural enrichment and interest, as do other pupils. But interest is primarily a means of stimulating effort, not a substitute for effort.

Principles of Arrangement. The following principles refer primarily to the *sequence* of the topics to be included in the curriculum.

(*1*) The sequence in the curriculum should be such that each topic will contribute definitely toward an ever-growing and more significant organization of the basic concepts, principles,

skills, facts, relationships, types of appreciation, and fields of application, resulting in the development of a unified mathematical picture.

(*2*) Even in a reduced program, the study should emphasize problem solving and modes of thinking, and should not become a mere sequence of formal and relatively abstract drills.

(*3*) If a unit organization is followed, it is not always advisable to attempt in each of the units a complete or exhaustive treatment of the central theme or topic under discussion. On the other hand a unit should not include unrelated "odds and ends."

(*4*) In general, a new topic should not be introduced unless there is a sufficient background of prerequisite concepts and skills to permit unhindered concentration upon the new elements.

(*5*) A new idea or principle should not, as a rule, be introduced prior to the time at which it is needed or may be effectively applied.

MATHEMATICAL CATEGORIES AS A BASIS OF ORGANIZATION OF THE CURRICULUM

The Doctrine of "Centers of Interest" Elementary and secondary school systems are operating in many cases under different conceptions as to the best way to carry on the "educative process" One can with some propriety speak of the "old education" and the "new education" although such descriptions tend to over-simplify the problem Not all schools of fifty or seventy-five years ago were alike; many of them were not so narrow in offerings or as restricted in points of view as is asserted in some present-day writing. In education, as elsewhere, one must remember that a careful study of the past reveals not only faults unsuspected by some people but virtues associated by others only with the present. On the other hand, what is called the "new education" may have more conservative elements than is sometimes realized. This much, however, it seems safe to say by

way of generalization. The "old education" interpreted teaching as a process of transmitting or inculcating a relatively fixed body of skills, information, facts, behaviors, and habits that were considered necessary or useful to a person, both in "making a living" and in living with satisfaction. It tended to employ fixed curricula, though not without giving attention in varying degrees to the interests and abilities of its clientele through the use of electives or different curricula. The "new education," in contrast, tends to be much more "child-centered." It questions fixed curricula, even when provision is made for electives, and seeks to induce the pupil to reach out for such elements of information or training as may be in harmony with his own "needs and interest," both felt and unfelt.

Under the "new education" an effort is being made in some schools to build the curriculum primarily around significant "centers of interest" or "areas of experience," so broad in character as to anticipate potentially a large number of the educational needs of children. By this means it is hoped to insure both a set of desirable practical and cultural backgrounds and the possibility of favorable subjective reactions. Furthermore, it is hoped that through sufficiently expert handling of these backgrounds the school will succeed in transmitting the skills, facts, habits, and attitudes necessary to a successful life and well-rounded development Under the "old education" many of the most important of these facts and skills have been isolated, logically organized, and systematically taught Under the "new education," however, they are to be acquired much more informally in the course of experiences selected, at least in part, for reasons other than their fruitfulness as a means of transmitting predetermined skills, facts, and habits.

In the elementary school the "new education" has secured widespread endorsement. The secondary school is being subjected to great pressure to follow a similar course; that is, it is being asked to give up its "adult-organized," sequential subject matter courses in favor of broad, flexible "areas of experience"

that are assumed to appeal to the individual student. In this Report the concern is mainly with the question of the extent to which such a conception is feasible in the field of mathematics

Unsettled Status of This Educational Issue. The fact has been stressed that the mathematics of the elementary school is a comparatively closely organized and cumulative system of concepts, skills, facts, and relationships. A pupil cannot master a given unit in most branches of mathematics until he has acquired some understanding of and reasonable skill in the related earlier steps involved. If the basic equipment in mathematics is to be acquired by means of "centers of interest," as these are commonly understood, such as the farm, transportation, and consumption of goods and services, mathematics becomes only one aspect of such study. Today a crucial question of curricular theory centers around the extent to which the methods of the "new education" lead to mastery or understanding of fundamental subjects such as mathematics.

The evidence accumulated on the relative merits of curricula organized according to the newer and the older theories is far from conclusive, at least with respect to instruction in mathematics. The "new education" is still definitely in the experimental stage in the secondary school. There can hardly be any doubt that an exclusive dependence upon "centers of interest" as a basis for organizing the curriculum makes very difficult the application of the guiding principles discussed in this Report. In the absence of more convincing evidence in favor of one point of view over the other, this Commission has sought a basis of organization that adheres to accepted principles, yet incorporates modern views on the way the mathematics curriculum should be organized The Commission recognizes the desirability of correlated activities and a reasonable utilization of centers of interest pertaining to important aspects of modern life, but under present conditions it recommends that for other than experimental classes the curriculum be organized in conformity with principles outlined above and later in this Report.

Expressing the Mathematics Curriculum in Terms of Broad Categories. For the reasons suggested above, this Commission has found it desirable to outline a general plan of organizing the materials of instruction in secondary mathematics, for grades 7–12, in terms of two principles of classification. *First,* there is the subdivision according to major subject fields. I. The field of number and of computation. II. The field of geometric form and space perception III. The field of graphic representation. IV The field of elementary analysis (algebra and trigonometry). V. The field of logical (or "straight") thinking VI. The field of relational thinking VII. The field of symbolic representation and thinking *Second,* there is the subdivision of the fields into categories such as the following: I. Basic concepts, principles, and terms II Fundamental processes III. Fundamental relations. IV. Skills and techniques V Applications.

A curriculum that is constructed in conformity with such a plan has the following merits:

(*1*) It consists of parts which, separately and in combination, for many years have constituted the essential elements of successful mathematics work in the grades considered.

(2) It is flexible enough to make possible its use in schools representing virtually all the principal administrative types of organization, such as the 8–4 plan, the 6–3–3 plan, the 6–6 plan, and so on

(*3*) It is sufficiently adaptable to meet a large variety of local conditions and special needs

(*4*) It provides definitely for continuity of training with respect to the central objectives of secondary mathematics.

(*5*) It suggests and makes possible extensive correlation with related fields

In later chapters of this Report two such curricula are described, without any implication that other satisfactory programs are not possible. Indeed other well-considered and tested plans have appeared which seem to the Commission to be basically in agreement with those described here. The plans do

not restrict the freedom of the teacher as to methods of teaching, or types of motivation, or lesson organization.

ESSENTIALS OF A GENERAL PROGRAM IN SECONDARY MATHEMATICS

In the following pages there is submitted in broad outline a summary of those mathematical fields and types of training and of appreciation that are necessary in order that the pupil may meet the demands of modern life and may realize the desirable cultural contributions that have been discussed. It is the opinion of this Commission that the mathematics program of our secondary schools, in grades 7-12, should be built substantially on abilities and outcomes such as those suggested below.

I. The Field of Number and of Computation

(A Continuation of the Work of the Elementary Grades)

Basic Concepts and Principles. There should be a growing familiarity with the basic vocabulary and working principles of arithmetic. This involves (*1*) naming or identifying a concept when encountered, (*2*) giving an example or an informal explanation of the meaning of given terms, and, at a higher level, (*3*) developing formal definitions of terms that have a broad operational significance. Examples of such terms are the following:

(*a*) Operations: addition, subtraction, multiplication, division, rounding off numbers.

(*b*) Results: sum, product, difference, quotient, per cent.

(*c*) Relations: ratio, proportion, equality, increase, decrease.

(*d*) Applications: interest, discount, commission, rate, premium, profit, loss, average

In all teaching of secondary mathematics much attention should be given to a conscious grasp of the *principles* which underlie the fundamental processes of arithmetic Examples of such principles are the following:

(*a*) The numerator and the denominator of a common fraction may be multiplied or divided by the same (non-zero) number, without changing the value of the fraction.

(*b*) The order of the factors in a product does not affect the result.

Fundamental Skills. The pupils should develop such skills as the following:

(*a*) The ability to use the four fundamental operations with integers, common fractions, and decimal fractions, due regard being given to the learner's maturity.[5]

(*b*) The ability to use the principal units of measure in everyday life situations.

(*c*) The ability to read simple numerical tables in connection with the educational program as a whole.

Application [6] There should be a gradual and continuous development of the ability to recognize and use arithmetical facts, concepts, and principles in everyday life situations, wherever encountered, not merely for the study of pertinent numerical problems, but also for purposes of explanation and prediction.

This ability should be stressed constantly until its use becomes a habit.

Further Topics There should be a gradual growth in the pupil's understanding of the extended number system (negative numbers, irrational numbers, and so on). Attention may also well be given to:

(*a*) The ability to perform addition, subtraction, multiplication, and division by use of a computing machine.

(*b*) The ability to multiply, divide, square, and find square roots by a slide rule.

(*c*) The ability to exercise judgment in presenting numerical results of measurement or of computations based upon measurements. Due regard being paid to the learner's level of maturity, there should be developed gradually the habit of retaining an appropriate number of significant digits and using appropriate accuracy in computation.

II THE FIELD OF GEOMETRIC FORM AND OF SPACE PERCEPTION

Basic Concepts The ability to recognize at least the elementary geometric figures and terms involves (*1*) naming a figure when seen or presented, (2) sketching or drawing a figure to illustrate a term, and, at a higher level, (3) developing a formal definition of the basic terms. The following list suggests types of terms that should receive attention:

[5] It is understood that these processes are to be considered not merely in an abstract way, but in many concrete problem situations such as those involving percentage and other business and social applications. Accuracy should receive the main consideration, but a reasonable degree of speed must also be regarded as essential.

[6] The description given here will apply with appropriate and rather obvious modifications to fields II, III, and IV, and reference will be made back to it on later pages.

(*a*) Entire figures: rectangle, circle, triangle, square, trapezoid, parallelogram, rectangular solid, cube, cone, pyramid, sphere.

(*b*) Parts of figures: radius, diameter, diagonal, vertex, sides, chord, arc.

(*c*) Mensurational terms: length, area, volume, perimeter, and such units as inch, foot, square inch, cubic foot, centimeter, meter, degree.

(*d*) Positional relationships: parallel, perpendicular, vertical, horizontal, oblique.

(*e*) Terms involving comparison: greater, less, equal, congruence, similarity, symmetry.

(*f*) Incidence relationships: intersection, tangency, coincidence.

Fundamental Skills. The drawing, the measuring, and the basic construction of the common geometric figures are skills, the development of which should represent a continuous program accompanying the study of the fundamental geometric relationships.

Among the skills to be stressed are the following.

(*a*) The ability to use such common instruments as the ruler, graduated or ungraduated, the compasses, the protractor, and squared paper.

(*b*) The sketching, or drawing, and the construction of common geometric figures, either full size or to scale.

(*c*) The direct measurement of lengths and of angles.

(*d*) The determination by formula of such common areas as those of the rectangle, the square, the triangle, the circle, and the trapezoid.

(*e*) The determination by formula of such common volumes as those of the prism, especially the rectangular solid and the cube, and the cylinder

(*f*) A use of the technique of indirect measurement in simple field work

Elementary Geometric Facts, Properties, and Relations The pupil should acquire such abilities as the following·

(*a*) The ability to recall and to apply habitually fundamental metric relations or propositions, such as the following· The sum of the angles of any triangle is 180°; the Pythagorean relation.

(*b*) The recognition of relations resulting from varying positions of geometric figures, such as the possible intersection of lines, of circles and lines.

(*c*) The ability to recognize and to state simple functional relations resulting from changes in dimension or position, such as the change in the area of a square whose side is doubled

Discovery and Verification There should be a gradual and increasing development of the ability to discover and to seek means of

testing important geometric relationships An acquaintance with the major propositions resulting from this training should become a definite part of the student's mathematical equipment

Application See p. 63

III. The Field of Graphic Representation

Basic Terms and Concepts There should be a growing familiarity with the basic vocabulary of graphic representation. As in other fields this involves (*1*) naming or identifying a concept when encountered, (*2*) giving an example or informal explanation of the meaning of given terms, and at a higher level, (*3*) developing formal definitions of the more important terms. Examples of such terms are the following: ordinate, abscissa, axis, coordinate, distance, tangent, line, slope, locus, graph, symmetry, table, formula, scale, bar chart.

Fundamental Skills The pupil should be able to:

(*a*) Take tabular data and construct therefrom a graph with a suitable scale, properly titled and of appropriate type.

(*b*) Read a given graph, recognizing not only values at intermediate points, but also rates of rise and fall, and maximum and minimum values

(*c*) (Optional) Draw a line to "fit" data approximately linear

Application See p 63.

Further Topics. Some schools may find it feasible to extend the work in graphic representation into the field of elementary statistics, including the use of alignment charts (nomograms).

IV The Field of Elementary Analysis

The Basic Vocabulary and Working Concepts of Elementary Analysis The pupil should be able to. (*1*) name or identify a concept when encountered, (*2*) give an example or an informal explanation of the meaning of the basic terms, and, at a higher level, (*3*) develop formal definitions of such terms as have a broad operational significance. Among these are the following.

(*a*) Types of number: positive and negative numbers, fractions, irrational numbers

(*b*) Operations. addition, subtraction, multiplication, division, reducing to lower terms, finding square root, raising to a power.

(*c*) Structural terms: monomial, binomial, polynomial, coefficient, exponent, radical, similar terms.

(*d*) Functional terms. equation, formula, variable, dependence, table, correspondence, sine, cosine, tangent.

(*e*) Applications. average, rate of motion or of work, evaluation of a formula, approximation, per cent of error.

Fundamental Principles and Techniques. The pupil should be able to use the fundamental principles of algebra and elementary analysis involved in basic techniques and in related pertinent applications Examples of such principles are: the rule for the addition of similar terms, the rule for reducing fractions to lower terms, the laws of exponents. Illustrations of these techniques follow.

(*a*) The fundamental manipulative techniques.
(*b*) The ability to solve equations.
(*c*) The ability to make trigonometric reductions.
(*d*) The ability to solve triangles.

Application See p. 63.

Further Topics. Some schools may find it desirable to extend this field of work to include in the later years some work in the technique of differentiation, together with applications that are within the comprehension of the pupil.

V. The Field of Logical (or "Straight") Thinking[7]

While it has always been recognized that mathematics is essentially a mode of *thinking*, it has not been equally clear precisely how the types of thinking characteristic of mathematics are to be stressed in connection with the customary materials of instruction. Nor is there general agreement as to the degree of emphasis that should be given to training in mathematical thinking. Recent psychological investigations tend to prove, however, that the "transfer" value of any given school subject is largely a result of a conscious and persistent application, in everyday situations, of *generalized* concepts, pro cedures, and types of thinking. The ability to apply widely and habitually such potentially broad areas of training is not acquired suddenly, but is the result of a gradual process of growth Hence every opportunity must be used in the classroom, throughout the entire mathematics program, to cultivate the active interest of the pupil in the modes of thinking which we are here discussing. These considerations suggest the need of analyses of the kind given below That is, in this field of work we are concerned with aspects such as the following.

Basic Terms and Concepts A clear understanding of the meaning of the basic terms and the ability to recognize their actual occurrences and their bearings in life situations. Examples of such

[7] In connection with this topic reference can be made to Keyser, C. J, *Thinking about Thinking,* E P Dutton and Co., New York, 1926, and Fawcett, H P., *The Nature of Proof,* Thirteenth Yearbook of the National Council of Teachers of Mathematics, Bureau of Publications, Teachers College, Columbia University, 1938.

concepts are: assumption or postulate, proposition, converse, conclusion.

Fundamental Principles. A clear grasp and appreciation of the assumptions and principles on which the structure of mathematics rests. This involves considerations such as the following:

(*a*) A knowledge of the principles underlying the manipulative techniques of mathematics, such as those relating to order, grouping, distribution, and the like.

(*b*) The realization of the logical implications of related propositions, such as those involving a given theorem, its converse, its opposite.

(*c*) The realization of the economy resulting from such an organizing principle or assumption as that of *continuity*.

Fundamental Abilities. Such abilities as the following should be gradually developed:

(*a*) To recognize and formulate the assumptions underlying an argument.

(*b*) To recognize terms that require precise definition.

(*c*) To organize statements in a coherent logical sequence.

(*d*) To recognize the proposition under discussion and to realize when a conclusion has been reached.

(*e*) To discover common flaws not only in reasoning in mathematical and related fields, but also in areas inviting emotional bias or requiring propaganda analysis.

(*f*) To recognize the logical structure or plan of an extended series of propositions, or of a related group of discussions.

Application. A gradual and increasing development of the ability to manifest coherent, logical thinking in everyday life situations.

VI. The Field of Relational Thinking

Here we are concerned with the development of a growing ability to recognize, in everyday life situations, cases of quantitative relationships and of functional dependence. The importance of this type of training was stressed as follows in the Report of the National Committee on Mathematical Requirements (p. 12):

"The primary and underlying principle of the course should be the idea of relationship between variables, including the methods of determining and expressing such relationship. The teacher should have this idea constantly in mind, and the pupil's advancement should be consciously directed along the lines which will present first one and then another of the ideas upon which finally the formation of the general concept of functionality depends."[8]

[8] Such strong emphasis was quite justified at that time, but the present Commission has already commented upon the possibility of over-emphasis. (See p. 42.)

Basic Concepts. The ability to recognize, to name, and in simple cases to define fundamental terms should be gradually developed. Such fundamental terms include: constant, variable, independent variable, dependent variable, one-to-one correspondence, function, formula, table, value of a function, invariant relation, increasing, decreasing, maximum, minimum, associated data, ordered list, interpolation.

Fundamental Skills and Abilities. Among these skills and abilities are the following:

(*a*) To read tables of related values, including trigonometric and logarithmic tables.

(*b*) To evaluate formulas for assigned values of the independent variables

(*c*) To interpolate in tables and graphs.

(*d*) To construct simple tables, such as frequency tables, from raw data, and numerical tables from given formulas.

(*e*) To construct appropriate formulas from verbal statements.

(*f*) To determine the constants (in very simple cases) for empirical formulas to fit approximately a set of given data.

(*g*) To recognize functional dependence, and to select variables pertinent to a given problem

Application There should be a gradual and increasing development of the ability to recognize functional dependence as well as statistical associations which fall short of functional correspondence.

VII. The Field of Symbolic Representation and Thinking

The aim here is a gradual development of the ability to translate quantitative statements into symbolic form and conversely, and an increasing appreciation of the economy and the power resulting from the correct use of symbolic techniques

HABITS AND APPRECIATIONS IN THE MATHEMATICS CLASSROOM

The development of desirable attitudes, habits, and appreciations is an outcome of the procedures employed by the teacher and of the resulting classroom atmosphere. Lasting results of this type, however, depend largely on continued attention to their growth, and cooperation throughout the school.

Every phase of the mathematics curriculum, as well as all classroom procedures, should be scrutinized for the opportuni-

ties to develop a growing appreciation of the immense power of mathematics, of its record as a universal servant of mankind, of its cultural significance, and of its permanent place in the study of nature, in the sciences, in the practical arts, in business, engineering, and everyday life.

Types of Motivation; Supplementary Projects and Activities. In recent years encouraging progress has been made in the direction of stressing the appreciational and cultural aspects of mathematics. Many mathematics classrooms are being equipped with illustrative visual aids and interesting charts and pictures giving evidence of the important place which mathematics occupies in the modern world The literature bearing on this phase of mathematical instruction is being extended. The *Yearbooks* of the National Council of Teachers of Mathematics, monographs published by *Scripta Mathematica,* numerous articles in such journals as *The Mathematics Teacher* and *School Science and Mathematics,* and a growing list of special treatises, all furnish valuable assistance to teachers who desire to enrich their usual programs by contacts with motivating backgrounds. Some effective modes of interesting pupils in this domain are suggested in the following paragraphs.

Nature as a Museum of Form. In the Introduction, attention was called to the great influence of nature upon mathematics. Many geometric forms are either clearly seen in or suggested by what man observes about him, and a constant mindfulness of this fact is to be recommended to all classroom teachers

Historical References The history of mathematics should not supersede mathematics; it should supplement its study. It is especially effective in increasing the cultural value of the subject, and a carefully developed account of parts of the history of this old enterprise will give more meaning to both elementary and secondary curricula in mathematics. Furthermore, it is to be recalled that historical considerations often increase a purely abstract interest that one finds in some topic or some problem that has a special appeal to him.

Dramatizing the Role of Mathematics in Modern Life. Every effort should be used to make pupils conscious of the contributions of mathematics to human progress and to individual and public welfare. The close connection of mathematics with the practical arts, with technology, industry, business, commerce, science, and other important enterprises, should receive attention. Among the methods that may be suggested for this work the following have proved successful:

(*a*) *School exhibits.* A periodic display, in classrooms and school corridors, of materials that will stimulate interest in present-day uses of mathematics can be recommended as a helpful device, particularly if the pupils themselves have furnished the materials.

(*b*) *Assembly programs.* Some mathematical units or topics are suitable for presentation before the entire school, in a dramatized form Brief plays, especially when prepared by the pupils themselves, seldom fail to arouse enthusiasm and sustained interest

(*c*) *Mathematical films.* The number of commercially available films that have a bearing on mathematics, while still very small, will gradually become larger; and such visual aids assist instruction in mathematics as they have already helped in the field of science Schools should be encouraged to make simple mathematical films as classroom projects.

(*d*) *Source books.* Pupils should be encouraged to collect and to preserve, in individual source books, items that have a bearing on mathematics. Those who are not acquainted with this device are always surprised to discover how very wide is the range of the source book materials that can be assembled in the course of a single year by any class of normal ability.

(*e*) *A permanent mathematical museum* In some schools a good beginning has been made in the direction of creating permanent mathematical exhibits. The ideal plan is that of assembling these collections in a central building, preferably in a special "Hall of Mathematics."

(f) Supplementary mathematical projects. Many enrichment projects based on local interests and showing local uses of mathematics will suggest themselves to a resourceful teacher. Among these might be mentioned scheduled talks by leading members of the community who find daily use for specialized mathematical training, such as engineers, actuaries, and bankers. Sometimes trips to large industrial plants, or for the purpose of engaging in elementary surveying, will add interest and vitality to the subject.

Mathematical Clubs. In many schools it is possible, among other extracurricular activities, to organize at least one flourishing mathematical club. It may be advisable to have a club for the earlier years of the high school as well as one for the later years Such clubs serve the important purpose of giving a suitable forum to particularly superior or enthusiastic pupils. The study of mathematical recreations, of advanced topics, of historical backgrounds, as well as discussions, reports, competitive tests, and debates will influence noticeably the daily classroom routine. If awards are offered for outstanding work done by members of the club, an effective incentive is often given to the entire mathematics program of the school.

Mathematical Bulletins or School Papers. Another extracurricular activity which has created enthusiasm and has stimulated effort, is that of enabling the strongest mathematics pupils of a school or of the community to edit a school paper devoted entirely to mathematics Such a publication may also be utilized by the teachers and the pupils as a medium for the collection of community problems, for book reviews and abstracts of important articles, and for important announcements It may also contain the programs of local or regional mathematical clubs, and may explain and discuss the mathematical equipment demanded by typical vocations or professions. In short, the mathematical paper of a school or a group of schools can attempt to show in what way mathematics is indeed a "mirror of civilization."

CHAPTER V

ONE DISTRIBUTION AND ORGANIZATION OF THE MATERIALS OF INSTRUCTION, GRADES 7–12

"Having premised thus much, we will now detain those who like our bill of fare no longer from their diet, and shall proceed directly to serve up the first course of our history for their entertainment."

—HENRY FIELDING, TOM JONES

INTRODUCTION

IT HAS frequently been charged that mathematical instruction is too resistant to change, the case of geometry usually being cited by way of illustration. We are told that more than two thousand years have passed since Euclid wrote his *Elements*; then we are reminded that this very successful textbook was prepared for select and intellectually mature students and that the great teacher of Alexandria did not have in mind the requirements of mass education. It is indeed quite true that although educational problems today are fundamentally very different from those with which Euclid was familiar, his famous work has been the model on which geometries have been constructed until recently With such facts in mind it is hard to evade such questions as these Is this the best we can do? Did elementary geometry reach its ultimate perfection with Euclid? Is no further improvement possible?

It is no longer necessary to say anything either in defense or in criticism of Euclid, or to debate the question of whether or not we were actually guided merely by tradition in our methods of teaching geometry. The fact is that pupils are being introduced to geometry in ways that differ significantly from those

formerly used It is true also that extensive changes have taken place in the teaching of arithmetic and algebra as well Beyond this there have been efforts to unify the different branches of mathematics in order to bring related ideas together, as well as efforts to develop new methods of approach in the hope of being more successful in arousing pupil interest. For many of these salutary changes credit should be given to the Report of the National Committee on Mathematical Requirements of 1923.

Notwithstanding the desirable readjustments that have already been made in mathematical teaching, and the indication that others are sure to follow, it is to be expected that there will be a considerable degree of permanence in the materials commonly used in the mathematics classroom. Mathematics represents one of man's greatest achievements in things that endure, not merely from year to year, but from century to century.[1] Throughout its history it has grown by steadily adding new elements while discarding or repudiating very little that it already possessed.[2] New ways of doing things have been found, more pleasing and powerful methods have been devised, deeper insights have been achieved, and concepts that at first were obscure have gradually been clarified. This growth and improvement have usually been achieved, however, without a disquieting revolution that has left all mathematics unsettled and confused for a period Certain theorems of mathematics may, with the passage of time, satisfy us less or seem less interesting, but so long as we retain underlying assumptions unaltered they remain valid theorems It should not surprise if we find such characteristics of mathematics reflected in its teaching.

CONSIDERATIONS RELATIVE TO TIME ALLOTMENT

It is undesirable to regard any high school year prior to the last as a "terminal" year for mathematical study, and to crowd

[1] "The fashions of this world are in continuous change and I would concern myself with things that are abiding "—Goethe.

[2] "The mathematics are distinguished by a peculiar privilege that, in the course of ages, they may always advance and can never recede."—Gibbon

into it numerous unrelated topics simply to make sure that the pupils will have encountered them. Pupils need to meet important topics repeatedly, on successively higher levels. Maturity also is requisite for the proper understanding of portions of the subject. It is therefore necessary to have a mathematical curriculum that runs through several years according to a systematic plan.

One must, however, face the fact that in many school systems the requirement of mathematics does not now extend through as many years as, in the opinion of the Commission, are justified by the best ultimate interests of the boys and girls. Hence there is the urgent practical problem of bringing about an extensive election of further mathematical courses beyond the required subjects Success in this undertaking will depend mainly upon two factors· (1) The degree to which administrators aid teachers in giving pupils adequate information on the importance of mathematical study, both practically and culturally, (2) the extent to which teachers themselves succeed in making the courses so rich in meaning that pupils will have gained an actually real insight into the true nature of mathematics and an enthusiasm for it.

The Commission believes that mathematics should be required through the ninth school year, and beyond the ninth year in the case of competent students, with the exceptions previously mentioned. In arriving at a decision as to required courses, and courses that are recommended or are made elective, the Commission urges that administrators keep the following considerations in mind:

(*1*) There is an increasing use of mathematics in modern life, and a growing demand for persons who have had adequate preparation in the subject.

(2) All pupils should be made acquainted with the mathematics requirements of professions, of business and industry, and they should also be informed of the cultural and social bearings of present-day courses in secondary mathematics

(3) Serious handicaps often result from the omission of or the indefinite postponement of mathematical training.

(4) Irrespective of where actual requirements may end, no particular year should be described as a normal year for terminating mathematical study.

(5) The time a pupil should spend in the study of mathematics should be determined by the degree of his understanding, appreciation, and mastery of the material that modern life seems to demand.

(6) In order to insure a pupil's growth in mathematical power and appreciation, the mathematical program must be carefully planned as a whole.

(7) The basic mathematical work in any school year, for pupils of approximately the same ability, should be the same for those who expect to enter vocational fields in the near future and for those who will have an opportunity to extend their training through further academic work.

(8) Capable pupils should be given an opportunity to continue their mathematical studies throughout the entire secondary period and should be encouraged to do so.

THE PROBLEM OF THE RETARDED PUPIL

Every teacher has had convincing personal experience with the different rates at which pupils make progress and is aware of the varying degrees of mastery that they achieve by comparable efforts There are also many objective studies that bear upon the question, and that show factually the great range in mental ages of pupils whose chronological ages are the same. Such data as these make it difficult to defend the practice of a single program in mathematics for all high school pupils; appropriate differentiated curricula seem to be called for. Chapter VII of this Report will deal with both accelerated and slow pupils. At this point it seems desirable, however, to pay some attention to the question of remedial work designed to reduce the number of pupils who must be classed as slow-moving

throughout their secondary years.[3] The Commission believes that the best information now available gives validity to the following statements:

(*1*) Much more careful work is necessary in the elementary grades if we are to prevent a continuance of the widespread disabilities now prevailing in the secondary schools. Careful experiments in representative school systems indicate that retardation is due in a large part to removable causes.

(2) Retarded groups of high school age should be given a program of work that includes a careful strengthening of the foundations in reading and arithmetic, at least one semester being needed for such work.

(*3*) In the case of retarded pupils, it is impossible at any stage to crowd into a single year all the necessary remedial work and also the usual program for that year.

(*4*) If the remedial work is begun as early as the seventh grade, retarded groups can cover a major fraction of a normal program in both grades 7 and 8; but unless an adequate foundation in fundamentals has been created, little is achieved in mathematics.

The program suggested for grade 9 and the higher grades in the remainder of this chapter assumes that the foundation in arithmetic and geometry has been well laid

SUGGESTED GRADE PLACEMENT CHART

GRADES 7-12

The program proposed in this chapter as one feasible curriculum plan is set forth both by verbal description and in the form of a chart (printed as Appendix V). A chart is used because of the condensation that it allows, and because of the general comprehensive grasp that it affords of the program as a whole It

[3] In this connection attention may be called to an important study concerning a remedial reading program in the high school by Stella S Center and Gladys L Persons, *Teaching High School Students to Read* English Monograph No 6, National Council of Teachers of English, D Appleton-Century Co, New York, 1937

is hoped that the chart will help to give the student of education, the administrator, and the mathematics teacher an understanding of the extensiveness of an adequate program in secondary mathematics, and a consequent realization of the time allotment necessary for the proper study of the subject.

In describing a program by means of a chart, there is risk of seeming to imply a greater rigidity than is intended[4] This Report has already stressed the importance of flexibility, and it is possible to assign parts of the various fields shown in the chart to other school years than those indicated. *The plan is intended for pupils of normal ability who have had good training. It is not, furthermore, implied that all the work will be covered by all pupils.*

In the chart and in the outline the entire subject of mathematics is divided into fields somewhat different from those on page 61. The composite field of elementary analysis is broken into its two constituent parts, algebra and trigonometry. On the other hand, the former fields of logical thinking, relational thinking, symbolic representation and thinking are united into a general field entitled *mathematical modes of thinking*, along with which are also considered habits, attitudes, and types of appreciation. The history of mathematics appears separately, as does also the problem of correlated mathematical projects and activities. The work to be done in all eight of the fields now considered extends through every year of the program, each year being expected to deepen and extend the pupil's experience with the broad categories discussed in Chapter IV. In the earlier years much of the work in certain fields will be "informal," but by slow degrees, as the pupil's mathematical power develops, a transition may be made to the more formal methods characteristic of any scientific and comprehensive development of mathematics.

[4] Curriculum summaries for grades 7, 8, and 9 have been used to good advantage by Raleigh Schorling, in *The Teaching of Mathematics*, pp. 102 107, Ann Arbor Press. While his outlines are much more detailed than the descriptions in the present chapter, a comparison shows quite an agreement in grade placement of topics.

It is to be noted that the program outlined makes definite provision for continuity of development of the different fields of mathematics, in accordance with the principles of Chapter IV. No discussion will be given of ways in which an integration of the different subjects is to be effected, for this is largely a matter of methods of teaching, a subject which this Report recognizes as of great importance, but cannot treat in detail.

THE WORK OF THE SEVENTH AND EIGHTH GRADES

The mathematical programs of the seventh and eighth grades are treated as a unit, and by way of preface some explanatory statements will be made concerning the roles in these grades of the chief mathematical subjects.

Arithmetic. On page 54 the Commission indicated what mathematical equipment it considers normal for the adequately prepared pupil who is entering the seventh grade. Experience has shown, however, that the work of the elementary grades often does not provide a sufficient mastery of the fundamental processes of arithmetic,[5] either for the needs of life or for further progress in mathematical study. Time accordingly must be devoted to further instruction in arithmetic, special attention being given to the development of insight as well as accuracy and speed.

In the applied phases of arithmetic there is no reason for a rigid adherence to any one sequence of problems, so long as adequate provision is made for the definite inclusion of types that are socially necessary; such problem backgrounds may center around significant projects or be organized in a more conven-

[5] The situation would be much worse if the doctrine of "stepping up" the teaching of arithmetic in the grades should become prevalent. The proponents of this doctrine assert that many children are not mature enough to profit from a formal study of arithmetic until they reach the third or even the fourth grade. The Commission, however, sees no compelling reason for subscribing to the proposal for "stepping up" arithmetic, but it recognizes that until definite and authoritative policies relating to this question have been adopted, there will be uncertainty in the minds of administrators as to the amount of mathematics that can be covered in grades seven and eight.

tional way There seems to be a growing belief that detailed work in such fields as banking, investments, taxation, and insurance should come in later grades Brief and informal treatments of such topics may, of course, be included in the work of normal eighth grade pupils.

Informal Geometry.[6] Work of an informal kind in geometry should be given recognition Some teachers believe that about as much time should be devoted to it in grades 7 and 8 as to arithmetic. There should be continuous attention to the work, for little good results from occasional unrelated lessons in geometry. Care in teaching the fundamental concepts is particularly needed, and teachers should not encourage the mere memorizing of definitions or the imitative construction of the basic figures. It must be recognized that the skills, the understanding, and the appreciations that are the desired outcome of the study are acquired only slowly. The time allowed at present sometimes seems insufficient

Graphic Representation.[7] The picturing of numerical data and the interpretation of graphs are now well established parts

[6] Informal geometry as the Commission conceives it includes the following:

1 Intuitive geometry. the pupil looks at a figure and says that something is obvious, as, for example, the proposition, "If two straight lines intersect, the vertical angles are equal"

2 Experimental geometry of a simple type, such as cutting out a paper triangle, tearing off the angles and then placing them adjacent to show that their sum is 180°.

3 Observational geometry, which consists in recognizing objects in the world about us as typifying certain geometric forms, and in seeing certain relationships that exist between them

4. Geometric constructions, leading to the acquisition of certain skills, such as bisecting an angle.

5. A simple approach to demonstration through informal proofs of such theorems as "The sum of the angles of a quadrilateral is 360°," after the pupil knows that the sum of the angles of a triangle is 180°.

[7] Strictly speaking, any visual representation of either numerical or spatial data may be included under the heading of "graphic representation" When thus interpreted, the drawing of geometric figures, scale drawing, mechanical drawing, map making, and similar enterprises, are aspects of graphic representation, an interpretation now used extensively in European schools In America, when one speaks of graphic representation, he refers usually to the picturing of numerical or statistical data

of mathematical instruction, but some cautions are to be observed. An excessive enthusiasm for graphic representation can lead and sometimes has led to graphs being used beyond the point where benefit results.[8] There should not be an indiscriminate employment of unorganized numerical data, selected solely because they are available. The teacher should give preference to data which are within the comprehension of the pupils, have real educational significance, and appeal to the learner's personal interest and imagination. It should be remembered that the pupil's ability to deal with convenient scales usually develops rather slowly.

Algebra The algebra that has been successfully introduced into grades 7 and 8 up to the present time has been limited largely to the understanding of the basic concepts, to the evaluation of formulas, and the solution of very simple equations. It seems possible and also desirable to include other algebraic material, but, if it is to prove effective, the work should be carefully planned and should be so organized as to be significant in itself as well as designed to furnish a good foundation for later algebraic study.

Trigonometry. In the plan here given no formal work in trigonometry is recommended for grades 7 or 8. It is well to point out, however, that during these two years the program outlined includes important concepts, skills, and factual knowledge which taken together constitute a necessary preparation for effective work in trigonometry. Instances are: such concepts as those of ratio and proportion, such skills as those involving the direct measurement of lengths and of angles, the technique of scale drawing, a familiarity with certain basic geometric propositions and relations, and the reading of mathematical tables.

Modes of Thinking, Habits, and Attitudes. The scope of instruction and the types of experience for the pupil which may be included under this heading are so extensive that it is not

[8] In *Handmaiden of the Sciences*, pp 90-91 (Williams and Wilkins Co., 1937), Eric T Bell describes picturesquely, that is, graphically, how a "plague of graphs" has descended upon us in recent years

expedient to give a detailed treatment here, there is, moreover, little general agreement as to the most effective ways of bringing about desired results. The Grade Placement Chart suggests some of the objectives which may be classified under the headings above. The extent to which the topics should permeate the entire course or may become the objects of direct and separate study by the pupils is admittedly an open question. For more extensive discussion of the general theme the reader is referred to current and pertinent literature.[9]

Historical Backgrounds During the seventh and eighth school years the historical background of mathematics should be drawn upon freely, for it gives an approach that is natural and interesting There are at present numerous books and monographs that can serve as the necessary sources.

Correlation with Life Situations and Other School Activities. No fixed plan will be suggested here as to the inclusion of suitable fields of application for the mathematical program. Naturally, the applied phases of this work will vary in accordance with the maturity level of the pupils, the total time allotment, community needs and interests, and the life and the spirit of the school as a whole. Definite provision should be made, however, for the study of many life situations that are mathematically rich and also for school activities that have a mathematical bearing. A number of appropriate fields of application for this stage of the pupil's development are indicated in the Grade Placement Chart.

The Detailed Outline In the detailed outline that follows, the numerals 7 and 8 refer to grades 7 and 8, respectively. When both numbers appear together, the indicated aspects of the curriculum are intended to extend throughout the two-year period.

As previously indicated, *the Commission recognizes that other satisfactory programs are possible.* It believes, however, that the

[9] This literature includes books on method, *Yearbooks* of the National Council, articles in *The Mathematics Teacher*, in *School Science and Mathematics*, and the like.

one here presented is practicable and has many desirable features. Parts at least have probably received as much validation as is possible for a program that is to be used in widely different schools.

Arithmetic

I. *Basic Concepts and Principles.* (7, 8)

(*1*) Development of a reasonable familiarity with the working vocabulary of arithmetic

- (*a*) Terms used in the fundamental techniques, such as sum, addend, multiplier, product, per cent.
- (*b*) Terms used in the applications of arithmetic, such as profit, loss, discount, interest.
- (*c*) Terms used in connection with the employment of the common units of measure.

(*2*) Making sure of a clear understanding of the basic principles of arithmetic, such as the following:

- (*a*) Dividend = divisor times quotient (plus remainder).
- (*b*) Fundamental principles of fractions.
- (*c*) The value of an exact decimal is not changed by annexing zeros at the right of the decimal.

II. *Basic Skills or Techniques.* (7, 8)

(*1*) The four fundamental operations, involving (*a*) whole numbers, (*b*) fractions, (*c*) decimals

(*2*) The skills and processes needed in dealing with percentage problems and with other business or social applications of arithmetic.

(*3*) The ability to use readily the tables of measure commonly needed in life situations, and to read such other tables as are commonly used in basic fields of elementary mathematics.

III. *Using Arithmetic in Problem Situations* (7, 8)

(*1*) Development of a problem-solving attitude

(*2*) Development of the ability to analyze arithmetical problems and to prepare a complete solution in written form.

(*3*) Continued study of suitable practical problems of increasing difficulty, such as the following.

- (*a*) Numerical problems arising in the pupil's immediate environment (the home, the school, the store, the community). (7)

(*b*) Everyday business problems (buying and selling, profit and loss, discount, commission, simple cases of interest). (7)
(*c*) Business or social problems demanding greater maturity (banking, investment, taxation, insurance). (8) (Preliminary informal treatment in grade 7, in superior classes)
(*d*) (Optional.) Problems arising in science, in the shop, in the household arts (7, 8)

Informal Geometry

I *Basic Concepts* (7, 8)

(*1*) Development of a reasonable familiarity with the working vocabulary of geometry.
(*2*) The ability to explain the meaning of certain key concepts used in grades 7 and 8, such as circle, angle, triangle, isosceles triangle, rectangle, square, perpendicular.
(*3*) Development of the realization that physical measurement and drawings are only approximate

II. *Basic Skills or Techniques* (7, 8)

(*1*) Learning to use the ruler, the compasses, the protractor, and squared paper, as geometric instruments. (7) (Other instruments, such as those needed in mechanical drawing, may be introduced whenever they serve a useful purpose.) (7, 8)
(*2*) Learning to sketch or draw the basic figures of geometry (7)
(*3*) Constructing (*a*) an angle equal to a given angle; (*b*) parallel lines, (*c*) an equilateral triangle, given a side; (*d*) an isosceles triangle, given the base and one leg; (*e*) related figures. (7)
(*4*) Constructing (*a*) perpendiculars; (*b*) bisectors, (*c*) common rectilinear figures, such as rectangles, squares, right triangles, regular hexagons, regular octagons, (*d*) related figures. (8)
(*5*) Constructing figures congruent or similar to a given figure. (8)
(*6*) Measuring lengths directly (English units and metric units). (7)
(*7*) Measuring angles directly. (7)
(*8*) Learning to estimate roughly the approximate accuracy of a measurement. (7, 8)
(*9*) Finding the perimeters of figures. (7)
(*10*) Finding the area of (*a*) a square; (*b*) a rectangle; (*c*) a triangle, (*d*) a parallelogram; (*e*) a trapezoid; (*f*) a circle; (*g*) irregular figures. (7, 8)
(*11*) Finding the volume of (*a*) a cube, (*b*) a rectangular solid; (*c*) a prism; (*d*) a cylinder. (7, 8)
(*12*) (Optional.) Finding the volume of a sphere, a cone, a pyramid. (8)

(*13*) Solving applied problems involving lengths, perimeters, areas and volume. (7, 8)

(*14*) Drawing figures to scale. (7, 8)

(*15*) Determining distances or angles indirectly by using (*a*) the method of scale drawing; (*b*) the method of congruent or similar triangles; (*c*) the Pythagorean relation; (*d*) the tangent ratio. (7, 8)

III. *Important Geometric Facts and Relations* (7, 8)

(*1*) Introductory study of the geometry of shape, of size, and of position, for purposes of motivation and orientation. (7)

(*2*) An introductory study of such basic figures as the circle, the angle, and the triangle, involving (*a*) the meaning of related terms, (*b*) measurement; (*c*) classification of angles and of triangles; (*d*) the informal study of important properties. (7)

(*3*) An informal study of symmetry as a basis for constructions. (8)

(*4*) An informal study of congruence and of similarity (8)

(*5*) An informal study of the Pythagorean relation. (8)

(*6*) A resulting acquaintance with such geometric facts, properties, and relations as may readily be derived by informal methods, and as have significance in the program of geometry as a whole Among these are:

(*a*) Important metric facts, such as the following.
- Radii of a circle are equal (7)
- All right angles are equal. (7)
- The sum of the angles of a triangle is 180°. (7)
- In the case of parallel lines, certain sets of angles are equal. (7)
- The angles of an equilateral triangle are equal. (7)
- The base angles of an isosceles triangle are equal (7)
- The Pythagorean relation. (8)
- Common rules of mensuration (7, 8)

(*b*) Certain positional relations, such as the following
- A line can intersect a circle in but two points (7)
- Two unequal circles may be in one of six possible positions with reference to each other. (7)
- Given a network of squares and two axes, the position of a point on that network may be determined by a pair of numbers. (7, 8)

(*c*) Important functional relations, such as the fact that
- If the diameter of a circle is doubled, the circumference is doubled. (7, 8)
- The area of a triangle depends on the length of the base and of the altitude (8)
- If the length of a rectangular solid is doubled, while the

other dimensions remain unchanged, the volume of the solid is doubled. (8)

IV. *Discovering and Testing Geometric Relationships.* (7, 8)

(*1*) Through the use of a laboratory technique, the pupil should begin to discover, to verify, and to analyze geometric relationships This may involve a variety of approaches, such as (*a*) inspection of drawings, (*b*) measuring; (*c*) superposition; (*d*) simple deductive analysis. (7, 8)

Graphic Representation

I Ability to interpret pictorial or graphic charts within the comprehension of the pupil. (7)

II. Making bar graphs based on simple statistical data having educational significance. (7, 8)

III Making circle graphs, preferably in connection with a study of budgets, of taxation, and related fields. (7, 8)

IV. Making line graphs to show change or fluctuation. (7, 8)

V. Making graphs based on social and economic data. (8)

Algebra

I. *Basic Concepts* (8)

(*1*) Development of a reasonable familiarity with pertinent algebraic terms used with the work of grade 8.

(*2*) Ability to explain clearly the meaning of certain key concepts, such as *coefficient, equation, formula, similar terms*

(*3*) (Optional.) Signed numbers.

II. *Basic Skills and Techniques.* (8)

(These should be based on an understanding of carefully considered *principles.* Only work in simple *monomials* should be stressed in grade 8.)

(*1*) Symbolic representation of simple quantitative statements or relations.

(*2*) Combining similar terms by addition or subtraction.

(*3*) Evaluating algebraic expressions or formulas. (This work may be begun in grade 7.)

(*4*) Solving equations, each involving either one or two steps.

(*5*) Reading a table of square roots.

(*6*) Making a formula as a shorthand expression of a mathematical rule (This work may be begun in grade 7)

(*7*) Making a formula representing a mathematical relationship
(*8*) (Optional.) Reading a table of tangents
(*9*) (Optional) Interpreting and using signed numbers in life situations.
(*10*) (Optional.) Making a table based on a stated relationship
(*11*) (Optional.) Interpreting a table of related number-pairs

III. *Using Algebra in Life Situations or in Problem-Solving* (8)

(*1*) Solving verbal problems by appropriate methods, including (*a*) table; (*b*) graph, (*c*) formula; (*d*) equation.

(2) Using the equation or the formula in the solution of common problems arising (*a*) in business; (*b*) in the shop, (*c*) in science, (*d*) in everyday life

Trigonometry (7, 8)

I. Familiarity with such terms as ratio, proportion, scale, congruent, similar, height, distance, horizontal, vertical.

II. Drawing a figure to scale on either unruled or squared paper

III (For superior pupils) Making simple outdoor measurements of lines or angles and thus finding required heights or distances

IV (For superior pupils) Using the "shadow method," or a table of tangents, to find unknown heights. (8)

V. (For superior pupils.) Making simple surveying instruments.

The Remaining Fields

The Grade Placement Chart gives suggestions for the other three fields that are shown on it

THE WORK OF THE NINTH GRADE

Since the ninth year occupies a special place in the rather prevalent 8–4 and 6–3–3 plans, it is natural that there should be lack of agreement in the different educational programs used for that year. The Grade Placement Chart, which summarizes the plan of the present chapter, presents the ninth grade program as one of general mathematics, with algebra serving as the central theme In this connection it may be mentioned that a large number of texts bearing the title *algebra* contain essentially the material that appears here under the designations *arithmetic, graphic representation,* and *trigonometry.*

Arithmetic. Emphasis on accurate arithmetic work should continue throughout the ninth year. Effective practice should be given in rounding off numbers to a desired number of significant digits, or of decimal places Opportunity for some computation arises in algebra in connection with such topics as the evaluation of algebraic expressions or formulas, the solution of equations, checking of the fundamental algebraic operations, the solution of problems, and the use of tables.

Geometry The abilities acquired during the seventh and eighth grades can now be used in illustrating, motivating, and applying algebra, in the solution of geometric problems; in scale drawing; in the study of indirect measurement; in (optional) a unit of demonstrative geometry.

Graphic Representation. The principal techniques used in making statistical graphs may be applied in various, more complex situations. Greater attention should be given to the graphic study of formulas and equations.

Algebra. By a wise and thoroughgoing elimination of unduly complicated technical processes, time has been gained in recent years for greater stress on understanding, on mastery of essentials, and on significant applications.[10]

[10] This improvement is probably due in large measure to the Report of the National Committee of 1923, which made the following statement that should still be kept in mind·

"Continued emphasis throughout the course must be placed on the development of ability to grasp and to utilize ideas, processes, and principles in the solution of concrete problems rather than on the acquisition of mere facility or skill in manipulation The excessive emphasis now commonly placed on manipulation is one of the main obstacles to intelligent progress. [The situation in this regard is acknowledged to be much better at the present time.] On the side of algebra, the ability to understand its language and to use it intelligently, the ability to analyze a problem, to formulate it mathematically, and to interpret the result must be dominant aims. *Drill in algebraic manipulation should be limited to those processes and to the degree of complexity required for a thorough understanding of principles and for probable applications either in common life or in subsequent courses which a substantial proportion of the pupils will take.* It must be conceived throughout as a means to an end, not an end in itself. Within these limits, skill in algebraic manipulation is important, and practice in this subject should be extended far enough to enable students to carry out the essential processes accurately and expeditiously."

Trigonometry. During the ninth year a serious beginning can be made in the teaching of the trigonometric ratios A unit of numerical trigonometry, making use of approximate values for the sine, cosine, and tangent, can be given.

Modes of Thinking and Desirable Habits and Attitudes. As the Grade Placement Chart suggests, the training under this heading begun in grades 7 and 8 is continued into the ninth, where it is possible to make an important extension by placing definite emphasis upon the functional concept If teachers will be on the alert to take the many opportunities which present themselves, they may reveal to their pupils the important place of mathematics in modern life, and do much toward creating desirable attitudes and appreciations through mathematical study

Historical Backgrounds. As far as time permits, historical references and comments should be interwoven with the topical developments of the year. This should not be done in a perfunctory way, but in a manner that shows the development of our civilization and the great part that mathematics has played in it Such instruction is of cultural value and assists in arousing interest Suggestions are given in the Grade Placement Chart.

Correlated Mathematical Projects or Activities. A moderate amount of time should be devoted to systematic study of the fundamental and extensive part that mathematics plays in contemporary life Much of this work will be of a supplementary character, and the "committee technique" is suggested as a means of collecting data, making displays, and giving significant reports

The Detailed Outline. It is of course assumed that modifications of the outline which follows will be made in order to adjust it to local conditions, or to the needs, interests, and preparatory backgrounds of pupils.

Arithmetic

I. Continuing the study of arithmetical concepts, skills, and principles, primarily in connection with the work in algebra

II. Solving applied problems, as suggested by the work in algebra, also problems involving approximate computations; the slide rule (optional).

Geometry

I. Reviewing and applying the geometric training suggested for grades 7 and 8, in connection with the work in algebra.
II. In classes of superior ability, making a beginning in the study of demonstrative geometry.

Graphic Representation

I. Reviewing and applying the techniques acquired in grades 7 and 8, in the representation of statistical data demanding greater maturity, mainly from the fields of business, economics, the social studies, and science.
II. Making graphs representing algebraic formulas, such as $p = 3s$, $i = .04p$, and so on.
III. Using graphs in the study of linear equations.
IV. (For superior pupils.) Using graphs in the study of simple cases of quadratic equations.

Algebra

I. *Basic Concepts.*

(*1*) The acquisition of the basic vocabulary.
(*2*) Developing the ability to explain clearly the meaning of key concepts, such as exponent, positive, negative, ratio, proportion.

II. *Fundamental Skills and Techniques.*

(These should be based on an understanding of carefully considered *concepts* and *principles*. Work with polynomials should be restricted to very simple cases.)

(*1*) The four fundamental operations, involving
- (*a*) Positive and negative numbers.
- (*b*) Algebraic monomials or simple polynomials.
- (*c*) Algebraic fractions, mainly with monomial denominators.

(*2*) Special products and factoring, as follows:
- (*a*) Squaring a binomial.
- (*b*) Finding the product of the sum and the difference of two terms.
- (*c*) Factoring a polynomial the terms of which contain a common monomial factor.
- (*d*) (Optional.) Factoring trinomials of the form $x^2 + bx + c$.

(*e*) Factoring the difference of two squares.

(*3*) Powers and roots.

(*a*) Laws of exponents and their use.

(*b*) Square roots of positive numbers.

(*c*) Fundamental operations involving radicals, mainly of the monomial type.

III. *Fundamental Principles.*

(*1*) A study of the principles governing the fundamental operations, such as the rules of order and grouping, the rules of signs and of exponents.

(*2*) A study of the principles used in the solution of equations, such as the rules of equality and of transformation.

IV. *Study of Relationships and of Dependence*

(*1*) By tables.

(*a*) Interpreting tables of related number-pairs

(*b*) Making tables based on formulas.

(*2*) By graphs

(*a*) Graphs as means of illustrating quantitative relationships.

(*b*) Making graphs based on tables of related number-pairs

(*c*) Making graphs based on algebraic expressions or formulas

(*d*) Using graphs in the study of equations

(*e*) Using graphs in the solution of problems.

(*3*) By formulas.

(*a*) Formulas as means of expressing relationship or dependence.

(*b*) Making formulas based on verbal statements, on geometric figures, on tables.

(*c*) Evaluating a formula

(*d*) Transforming a formula (only simple cases).

(*4*) By equations.

(*a*) Equations as means of expressing quantitative relationships

(*b*) Solving equations of the first degree in one unknown.

(*c*) Solving pairs of equations of the first degree.

(*d*) Solving fractional equations.

(*e*) Solving equations of the form $ax^2 = b$.

(*f*) Solving simple radical equations.

(*g*) Using equations in the study of proportion and of variation.

(*h*) Using equations in the solution of problems stated in verbal form

V. *Using Algebra in Life Situations and in Problem-Solving*

(*1*) Learning to translate quantitative statements into the language of algebra

(*2*) Learning to make generalizations suggested by the techniques and principles of algebra, particularly with relation to the precise way in which definitely related, changing quantities will influence each other under given conditions.

(*3*) Solving general verbal problems, using as a means of solution the table; the graph; the formula; the equation.

(*4*) Applying the techniques of algebra in problem situations arising in business; in the shop, in science; in everyday life.

(*5*) Interpreting the solutions of equations, including negative values, where they have significance.

Trigonometry

I. Reviewing the necessary concepts and skills.
II. Finding heights or distances indirectly by scale drawing; the Pythagorean relation.
III. Finding heights, or distances, or angles, indirectly by using the natural trigonometric functions (sine, cosine, tangent).
IV. Using a table of natural functions. (Interpolation should be regarded as optional)

The Remaining Fields

The Grade Placement Chart gives suggestions for the other three fields that are shown on it.

THE WORK OF THE TENTH GRADE[11]

We are concerned in the tenth grade with a plan that adds significant mathematical training but does not allow past instruction to lapse and be forgotten. Although demonstrative plane geometry will be regarded as the central theme for the year, the mathematical fields previously stressed are given attention, and there is ample opportunity for a considerable degree of correlation and integration of the new and the old. The out-

[11] The treatment of grades 10, 11, and 12 will be more condensed than that given for grades 7, 8, and 9, where a general discussion of the different fields was first given, followed by a detailed outline All that is said about any one field will now appear under a single caption.

line that follows is not exhaustive and it admits of either extensions or contractions.

Arithmetic. Facility in arithmetical computation should be stressed. Definite review and extension of the concepts of the subject in connection with applied problems are needed in most classes. The use of irrational numbers arises in connection with the Pythagorean theorem. Approximate computations occur in connection with trigonometric work, where the slide rule may be used.

Geometry. As stated above, the central theme of the tenth year is demonstrative plane geometry. In the case of pupils who have not had the benefit of a good course in informal geometry in the earlier grades, it is necessary to present in a few weeks a number of items that should have received attention earlier. Sufficient time must be allowed before the introduction of formal demonstration to permit the pupil to become acquainted with the basic concepts and to become skilled in the use of geometric instruments. Methods of formal demonstration must not appear on the scene too abruptly.

Standard lists of theorems have been issued by various examining bodies, such as the College Entrance Board and the Regents of the University of the State of New York. Of particular interest, also, is the list of propositions in plane and solid geometry prepared by the National Committee of 1923. Any one of these formulations may be regarded as sufficiently authoritative for purposes of classroom instruction, and no new list is considered necessary by this Commission in connection with the present plan.

Whatever list of theorems may be covered in the course, at least the following objectives should be set for all pupils

I. Understanding of fundamentals: (*1*) basic concepts; (*2*) the vocabulary of geometry, (*3*) the nature of geometric proof and its various forms, (*4*) the significance of undefined terms, definitions, and unproved propositions (postulates and axioms)

II Acquisition of skills, with an improvement of those set for grades 7, 8, and 9, and the addition of skill in demonstration

III. Familiarity with facts of geometry, both those discovered by informal methods in grades 7, 8, and 9, and those established deductively. The facts and principles should be organized clearly in the pupil's mind around certain central topics or themes, such as: (*1*) parallelism, (*2*) congruence; (*3*) similarity and trigonometric relations, (*4*) concurrence; (*5*) indirect measurement; (*6*) mensuration; (*7*) loci, and (*8*) construction They should also be organized around such basic figures as: (*1*) the triangle, (*2*) the parallelogram; (*3*) the circle, and so on

IV Development of elementary spatial insight, and the habit of noticing geometric relations in three dimensions, and of comparing such relations with those in a plane. (E g., Are non-coplanar angles equal, if formed by pairs of parallel lines? How many perpendiculars can be drawn to a line at a point? How should the angle between two planes be defined, and what question is raised thereby?) Such a development can be secured by discussion and generalization of certain propositions or problems of plane geometry.

V. Some realization of the significance of geometry in human affairs, both practically and culturally

VI. A clear grasp of what is meant by a deductive science; and ability to apply the method of postulational thinking in life situations not specifically mathematical.

Note. In the Grade Placement Chart, objectives V and VI occur under Mathematical Modes of Thinking, but they are so important that they are mentioned here under specific objectives for geometry.

Graphic Representation. In this grade the work in graphic representation may be restricted to a review of what has been done previously and to an extension in connection with the social studies and science programs. Drawing of graphs in connection with simple equations may be considered an optional topic.

Algebra. A purposeful interweaving of algebra and geometry may be accomplished by the following types of work: (*1*) the use of algebraic symbolism wherever it is of advantage; e.g., the use of letters to denote lengths, angles, etc., as $\frac{a}{b}$ in place of $\frac{BC}{AC}$; (*2*) algebraic proofs of certain theorems, such as that of Pythagoras; (*3*) theorems in mensuration in generalized form, such as the law of cosines; (*4*) algebra in the numerous geometric theorems and problems based on ratio and proportion; (*5*) simple locus problems; (*6*) equations involved in certain geometric

theorems. If their use is desired in trigonometry logarithms may be taught.

Trigonometry. The study of similarity and proportion furnishes a firm basis for the trigonometric functions The subject can be further related to geometry by deriving the exact values of the functions for 30°, 45°, and 60°, and by expressing the area of a triangle in terms of two sides and the included angle. The law of sines and the law of cosines may be introduced. Logarithms may be employed, but successful teaching of numerical trigonometry at this level is possible without them.

The Remaining Fields. The Grade Placement Chart gives suggestions for the other three fields that are shown on it

THE WORK OF THE ELEVENTH GRADE

The plan described in this chapter is compatible with the recent effort of the College Entrance Examination Board and other bodies to get away from a too rigid compartmental plan of teaching mathematics. It recognizes that, with new outlooks, mathematical curricula may be organized along broader lines than in the past, and at the same time pay regard to both continuity and flexibility. The Grade Placement Chart indicates that for the eleventh year in the present plan the central theme is the extension of algebra and a systematic study of elementary trigonometry. The two subjects are correlated through the use of the function concept as a unifying bond. Many of the topics of algebra begun in earlier years are extended. Reviews should be provided in these topics whenever they are needed

Number and Computation. A better understanding of the number system will result from the study of irrationals and an introductory consideration of imaginary numbers, which will form part of the more extended and detailed study of algebra Emphasis on approximate computation is suggested by the study of logarithms and especially by their use in the trigonometric solution of triangles The writing of numbers so as to employ indicated powers of 10 can be correlated with work in science.

Finally, the theory of the slide rule may be explained; computing machines, as well as various tabulating devices, may be discussed informally, primarily for informational purposes.

Geometry. No new work in geometry is contemplated, but there is occasion for considerable review in connection with the work in trigonometry.

Graphic Representation. Instruction in graphic representation can be carried on in connection with (*1*) a consideration of more complicated statistical data than had been used previously; (*2*) linear and quadratic functions; (*3*) solutions of equations; (*4*) graphs of the trigonometric functions.

Algebra. Throughout the work in algebra the function concept should be continually stressed. This can be done advantageously in connection with the study of tables, graphs, formulas, and equations. Important phases of the work in algebra are: (*1*) review and extension of basic concepts and operations; (*2*) a study of linear and quadratic functions, together with the solution of numerical and literal equations of the first and second degrees in one unknown; (*3*) the solution of sets of equations involving two unknowns, including simple sets of second degree equations; (*4*) a detailed study of exponents and radicals, with the solution of simple types of radical equations; (*5*) a study of logarithms and the slide rule; (*6*) the solution of simple types of exponential equations, such as those that arise in common problems involving compound interest; (*7*) a general study of formulas, their transformation, and, in simple cases, their graphic representation; (*8*) types of variation and their mathematical expression; (*9*) arithmetic, geometric, and binomial series; (*10*) if time permits, a brief introduction to statistical processes and formulas; (*11*) significant applications of the preceding topics.

Trigonometry. Suitable topics for inclusion in trigonometry are: (*1*) definition of the six functions for the general angle and the reduction formulas that occur therewith; (*2*) basic identities involving a single angle; (*3*) the addition formulas for the sine,

cosine, and tangent; (*4*) double-angle and half-angle formulas, (*5*) the law of sines, the law of cosines, and the law of tangents; (*6*) solution of triangles, including problems on heights and distances, (*7*) components and resultants; (*8*) work with simple identities and trigonometric equations; (*9*) field work. (Some of the work indicated may be postponed to the twelfth year, such postponement being especially desirable if the algebraic work of the ninth year did not include all the topics suggested for that year. Topics in trigonometry that could be deferred are the addition formulas, the double- and half-angle formulas, and the law of tangents.)

The Remaining Fields. The Grade Placement Chart gives suggestions for the other three fields.

THE WORK OF THE TWELFTH GRADE

Instruction for the twelfth year should make possible the rounding out of the mathematical picture developed during the earlier grades and at the same time give preparation for future work for those who will continue their study of the subject In the present plan, work for the year consists of a unified course from the fields of advanced algebra, solid geometry, analytic geometry, trigonometry, and differential calculus

Number and Computation The work includes (*1*) a review of the number system through irrationals, with emphasis on clear concepts and on an understanding of the laws of operation, (*2*) use of the polar as well as the rectangular form for writing and operating with complex numbers, (*3*) use of the derivative for the determination of approximate errors in computation.

Solid Geometry. Since the pupil now has his first opportunity for systematic study of three-dimensional geometry, the work should include the usual theorems and applications, particular attention being given to the relation of lines and planes in space, and to the properties and the measurement of prisms, pyramids, cylinders, cones, and spheres

The basic formulas of mensuration can be established in the

customary way, or by the use of such theorems as that of Cavalieri and the Prismoid formula. Much attention should be given to the visualization of spatial figures and relations, to the representation of three-dimensional figures on paper, and to the solution of problems in mensuration. The latter problems offer opportunity for correlation of solid geometry with arithmetic, algebra, and trigonometry Part of the importance of the geometry of the sphere comes from the new perspective that it makes possible

Analytic Geometry The analytic geometry included is designed to show the power and the generality of the analytic method of treating geometry and to give the pupil the basis that is needed for some of the work in analysis. The study should deal briefly with the straight line and the circle The work on the straight line should include the slope-point form and the two-point form, with the necessary preliminaries It should also embrace work on parallels and perpendiculars, with related applications The work on the circle should include the standard form for the equation and reduction to that form. Locus problems leading to straight lines and circles should be introduced when possible. It may also be possible to include a little work on the parabola and the ellipse

Graphic Representation. Further instruction in graphic representation will come through (*1*) graphic solution of equations; (2) representation of complex numbers, and (3) use of logarithmic paper (optional)

Algebra and Differential Calculus. The work from algebra and differential calculus embraces: (*1*) a study of rational integral functions of the *n*th degree, including synthetic division, the remainder theorem and the factor theorem; (2) relations between the roots and the coefficients of an equation; (3) Descartes' rule of signs, (*4*) determination of irrational roots; (5) permutations, combinations, and simpler cases of probability; (*6*) introductory study of differentiation, limited to polynomials, with applications to slopes, maxima and minima, rates of

changes, velocity, acceleration, and related problems. Topics from algebra that may be taken in addition to, or in place of some of those mentioned, are:[12] (*7*) a brief treatment of compound interest and annuities; (*8*) further work in statistics

Trigonometry. After a review of the work of the eleventh year trigonometry some of the following topics can be studied (*1*) radian measure; (*2*) periodicity of the trigonometric functions, with applications to oscillating physical quantities, (*3*) inverse trigonometric functions; (*4*) identities and trigonometric equations; (*5*) DeMoivre's theorem (Topics (*2*), (*3*), and (*5*) may need to be omitted if some of the work in trigonometry for the eleventh year is postponed to the twelfth)

History and Correlated Activities Supplementary readings, projects, and activities that require some maturity are within the reach of students during their twelfth year Such activities can contribute materially toward creating a lasting impression as to the nature and scope of mathematics

[12] It is to be noted that the College Entrance Examination Board's *Description of Examinations*, edition of December, 1938, says of Mathematics Gamma (the most searching mathematics examination offered): "The examination will contain no questions on determinants, simultaneous quadratics, scales of notation, or mathematical induction" In this Report these topics are suggested as possible extra ones for truly superior pupils, with a caution about including too much material at the expense of thoroughness See p. 146

CHAPTER VI

A SECOND CURRICULUM PLAN

"The teacher will find he can interest the pupil in it quite well by beginning with the old-fashioned four quarters of the globe, and coming round to the child's own parish by way of Africa and Zululand."

—MATTHEW ARNOLD, GENERAL REPORT FOR 1878

INTRODUCTION

VARIOUS arrangements of instructional materials are consistent with the broad principles set forth in Chapter IV, and it is not to be expected that any single type of curriculum will be accepted by all schools. Principles for curriculum construction must be interpreted in accordance with the conditions that affect the schools of a given community, and conditions may differ markedly from one locality to another. The plan outlined in the preceding chapter, perhaps with modified emphasis on some topics or with minor rearrangements, should suit the needs of many schools. On the other hand, substantially different plans, such as those evolved experimentally by various leaders in secondary mathematics, may be preferred by some administrators and teachers. It is, of course, undesirable even to attempt to outline all the different plans of organization that are consistent with the principles of curriculum building previously discussed, and that may be adapted to the needs of diverse school conditions or objectives. The Commission will, however, describe in this chapter a second plan for organizing the curriculum materials for grades 7 to 12.

The chief features of the present plan are as follows. In the ninth grade there is to be a course in general mathematics involving a more extensive consideration of arithmetic than in the

foregoing plan, both as to processes and as to applications. In the tenth year there is to be offered a course in demonstrative geometry which is quite comprehensive, but which should nevertheless appeal also to many pupils who do not plan to go to higher institutions. The eleventh year is to be devoted to a substantial course in algebra. In the twelfth year the pupil may select from a variety of courses, described later In each of the grades 10 to 12, important topics of earlier courses are reviewed, use being made of algebraic processes in geometry and geometric problems in algebra

In the case of larger schools the plan includes in addition during the ninth year a course in algebra, intended especially for pupils who expect to follow a profession that requires considerable mathematical training, or who prefer such a course In the tenth year such pupils will take the work in geometry referred to above. In the eleventh year a third term of algebra is to be provided for them, and a term of elective work is to be chosen from such courses as are available for the work of the twelfth year.

In the description of the work of the different years nothing is said as to modes of thinking, habits, and attitudes, of historical backgrounds, or of correlation with life situations and other school activities—topics discussed somewhat in the preceding chapter. What was said in that chapter, as well as the entries on the Grade Placement Chart relating to such matters, can again serve as a guide for the program that is here described

A few comments of a slightly pedagogic character are made in order to clarify some of the suggestions.

THE WORK OF THE SEVENTH AND EIGHTH GRADES

Arithmetic. The contemplated work in arithmetic, including the applications, involves only the more common concepts, principles, and techniques For example· finding the interest on a sum of money for a whole number of years, but not for a fraction of a year specified by two dates, simple discount, but not

the value of an interest-bearing note, discounted before due; the nature and functions of corporations, insurance, and taxes, but only simple calculations in connection with the problems that are given.

Geometry. The work in geometry is designed to give an acquaintance with the figures mentioned in Chapter V. It also gives some related numerical work, and instruction in the drawing of figures It does not, however, provide training in discovering or testing theorems, nor give practice in using relations that involve such topics as congruence, parallel lines, and special kinds of triangles

Graphic Representation. As described in Chapter V.

Algebra. The work in algebra is assumed to be limited to the writing of formulas and simple relations. Since many teachers feel that interest in algebra is not stimulated, but may be destroyed, if pupils are required to use algebra unnecessarily, it follows that, in a restricted course, the emphasis should be upon training pupils to express in symbolic language rules they are using rather than upon systematic substitution in formulas. Work with negative numbers—if included at all—is supposed to be limited to their meaning, and to their addition, with no attempt at teaching the other operations.

Trigonometry. Figures are to be constructed to scale, and proportions are to be used to find lengths, but no work with the trigonometric ratios as such is contemplated

THE COURSE IN GENERAL MATHEMATICS FOR THE NINTH GRADE

The work suggested here for the ninth grade is a composite course consisting of arithmetic, graphic representation, algebra, trigonometry, social mathematics, geometry, and logarithms. A brief description of the content of each of the subjects follows.

Arithmetic Fractions, mixed numbers, decimals, and per cents are to be used frequently, so that the pupil will increase his skills. Work on mensuration is to be reviewed and extended,

not merely as a way of giving computational practice, but also for the purpose of calling attention to geometric forms and their uses Units from the metric system are to be used Percentage is reviewed and extended, especially as a means of showing how the use of equations reduces the number of rules and techniques that must be learned Work in bookkeeping is suggested, even if only to a sufficient extent to show the nature of debit and credit entries in simple accounts Approximate numbers, significant figures, estimating results, and shortened methods of multiplying are included as preparatory to the work with trigonometric ratios Applications of arithmetic are to be extended to include such topics as per cent of error, medians, and weighted averages.

It is recommended that particular attention be given to social problems that include more steps than those encountered in the previous grades, even though each step be a simple one, emphasis being placed on the problem as a whole. For example: In earlier grades the pupil might find the cost of 10 tons of coal at $8 a ton, or the interest on $4,000 at 5% for 7 years. Such problems are unrelated. A typical ninth grade problem is: "What is the monthly cost of a house valued at $5,000 if money is worth 4%, if the insurance is 48 cents per hundred dollars, if the owner used 8 tons of coal at $9 per ton, if repairs amount to $102, and if depreciation at 2½% is allowed, all the expenses mentioned being on an annual basis?"

The number and variety of occupations has increased so greatly in the last decade that it is no longer possible to have the applications of arithmetic include samples from every type of activity. Regardless, however, of how the pupil will some day earn his living, he will always be a citizen and a consumer of goods and services Hence there should be an earnest effort to include in the instruction in arithmetic many problems based on activities and interests of the ordinary citizen Compared with practice in the past there should be much more work involving such topics as home-owning, mortgages, taxes, install-

ment buying, insurance, investments, automobile expenses, debts, risks, health, food, budgets, building and loan associations, cooperative enterprises, and the like. The mathematics teacher should use such topics not merely as material for computations, but because an understanding of them involves quantitative relations. He should be fitted to discuss many of them. The information that the pupil receives may be more important than the benefit to be derived from the computation. The Commission recognizes that the immaturity of the pupil will prohibit an exhaustive study of the topics suggested, any of which might be considered further in grade 12, as suggested later in this chapter

Graphic Representation. There can be work in the construction and interpretation of pictograms, broken line graphs, bar and circle graphs, and graphs of formulas of the types:

$$y = ax + b, \quad y = ax^2, \quad xy = k.$$

Simple cases of the converse problems of fitting such formulas to data can also be considered in some cases.

Algebra. The work in algebra that is suggested as part of the ninth grade course in general mathematics is very restricted, in so far as technique is concerned. It is centered around such aspects of algebra as the following:

(*1*) The use and interpretation of signed numbers.
(*2*) Algebra as a language, that is, the use of symbols to express ideas
(*3*) The meaning and importance of generalizing a problem and its solution
(*4*) The use of equations to solve problems which cannot be solved easily by arithmetic.

Instruction under (2) should include not only the writing of formulas, but also the writing in mathematical symbols, or a study of the meaning of such statements as: "When adding several numbers it is immaterial which numbers are added first." "The increase in profits was not proportional to the increase in sales." "To find a quotient correct to n places carry the division to $n + 1$ places."

As an illustration of instruction under (*3*) one can contrast the problem, "How many gallons of water must be added to 10 gallons of a 12% solution of an acid in order to change it to a 9% solution?", with the problem, "How many gallons of water must be added to g gallons of a t% solution to make it an s% solution?" Many teachers believe that such problems, slighted in the past because of assumed difficulty, illustrate one of the most important uses of algebra. By omitting less important work, it should be possible to give adequate instruction in such questions.

The complexity of the equations to be considered under (*4*) should be determined in general by the nature of the problems that are to be solved, though it is to be remembered that some work with equations of greater difficulty is usually necessary in order to make sure that methods are well learned and adequate skill is acquired. A consideration of problems suitable for the course, from which the "answer known" type is eliminated, will show that the equations are not more complicated than:

$$2\,4 = 1.2x, \qquad .0932y = 1.67, \qquad n - 8n = 4\,60,$$

$$\frac{3}{4}a - 5 = \frac{1}{2}a, \qquad .03p + 06(4500 - p) = .05(4500),$$

$$\frac{r}{5} = \frac{7}{12}, \qquad h^2 - 16 = 100, \qquad 5w^2 = 13$$

The literal equations which arise in the problems of generalization are not likely to be more complicated than

$$nx = s(x - a), \quad ax + b(s - x) = cd, \quad a = \frac{b - x}{x}$$

In the equations above no algebraic technique is involved beyond multiplying, as in $b(a - x)$; factoring such an expression as $ax + bx$, and knowing that the statement $nx = a$ implies that $x = a/n$, where a and n may be any quantities, provided $n \neq 0$. It is not expected that the solutions of literal equations will be checked by substitution, for such an operation may involve more algebraic technique than is contemplated.

The subject of ratio and proportion should not be slighted, but should be used extensively in connection with similar figures, or for comparing two areas, two volumes, two gears, or two levers, as a means of arriving at some desired relation. This work includes a study of the connection between proportions and direct and indirect variations.

The study of dependence will be made more fruitful by emphasizing that an equation such as $L = 2t + 1$ is not merely a formula for L, showing how L changes as t changes; but is a relation between two variables, from which either can be found when the value of the other is substituted.

It is assumed that exponents will be used chiefly in stating formulas that involve squares, cubes, etc., and in writing numbers in standard form, such as:

$$45{,}600 = 4.56 \times 10^4, \quad .0000456 = 4.56 \times 10^{-5}.$$

The work on radicals is supposed to be limited to using a table of roots of integers from 1 to 100 for instance and making such changes as:

$$\sqrt{240} = 4\sqrt{15}, \quad \sqrt{\frac{5}{8}} = \frac{1}{4}\sqrt{10}, \quad \frac{3}{\sqrt{5}} = \frac{3}{5}\sqrt{5}.$$

The arithmetic process of finding a square root should be reviewed; interpolation should be studied; the Pythagorean relation should be used, but only experimental testing of it need be considered, since even the simplest proof involves a more thorough study of triangles than is contemplated in the geometry work; the formula for the area of a triangle in terms of its sides (Heron's formula), and other formulas involving square roots, may be included.

To avoid misunderstanding it is to be noted that the proposed algebraic work does not include systematic study of the following:

(*1*) Multiplications of the types

$$4x^2 \cdot 5x^3, \qquad (x - 2)(x + 5)$$

(2) The four operations with fractions.
(3) Factoring, except that of $ax + bx$.
(4) Sets of equations, of quadratic equations, except equations such as $d^2 - 16 = 100$, and $5x^2 = 13$
(5) The "answer known" type of problem involving ages, coins, time-rate-distance, etc

It is of course not implied that teachers should not go beyond what is outlined if the preparation of the pupils and their abilities and interests warrant it

Trigonometry As a basis for work in trigonometry there should be scale drawings, with consideration of similar triangles and ratio of similitude The three ratios—sine, cosine, and tangent—are to be defined, and instruction is to be given in the use of tables to four decimals, with a tabular interval of 1°, interpolation being to a tenth of a degree. It is not contemplated that relations between the functions be studied, except the relations $\cos A = \sin(90° - A)$, $\sin A = \cos(90° - A)$

Other Topics. As mentioned elsewhere in this Report, opportunity should be taken to give the pupil an appreciative knowledge of the historical background of mathematics, and efforts should be made to develop those attitudes and habits of thinking discussed in Chapter III As observed earlier in this chapter, suggestions made in Chapter V can be utilized. In addition to such work, some classes are able—if too much time is not required to strengthen the arithmetic work of previous grades—to include one or both of the following topics

(1) Geometry When this study is undertaken, the chief objects should be (*a*) to impart certain information about geometric forms, and (*b*) to illustrate the nature of postulational thinking. The two objectives should not be separated; each week's work should show progress toward both goals. Assuming that the work is spread over fifteen or twenty lessons, the second objective can be reached by presenting a sequence of theorems, each depending upon and derived in a logical manner from some previous general statements It is, for example, possible to begin with congruent triangles and advance to a proof of such theorems as: "The base angles of an isosceles triangle are equal," "The sum of the angles of a triangle is 180°," or

some of the theorems about parallelograms. [illegible] are quite numerous.

(2) Logarithms. When this subject [illegible] be carried so far that four place tables [illegible]. Many teachers believe that such work is of more [illegible] pupils than much of the algebra which [illegible]. In order that pupils who have [illegible] might continue to use them, it would be [illegible] in mathematics contained a table of [illegible] then study of logarithms, pupils [illegible] slide rules, even though they [illegible]

Remarks. The general mathe[illegible] may be characterized as follows.

(1) It aims to correct those [illegible] may arise because some of the work [illegible] deferred to later years.

(2) It provides the training in [illegible] tion, algebra, geometry and [illegible] pupil will need while still in high [illegible] physics, chemistry, economics and [illegible]

(3) It affords a fairly broad mathe[illegible] look; and it is suitable for all pupils [illegible] equipped to offer two distinct mathe[illegible] ninth grade.

THE COURSE IN ALGEBRA FOR THE [illegible]

The course in algebra that the [illegible] larger schools, in addition to the [illegible] can include such topics as the following:

(1) Equations:
- (a) Linear equations with fractional [illegible]
- (b) Literal linear equations
- (c) Sets of linear equations in [illegible]
- (d) Sets of literal linear equations [illegible]
- (e) Quadratic equations solved [illegible] the square, and by the formula. [illegible] tional, checking may be omitted.
- (f) Fractional equations leading to linear [illegible] tions.

(*g*) Radical equations not more complicated than $2\sqrt{x} = 3$.
(*h*) Dependence and variation, including work on finding an equation from data concerning the variables

(*2*) Fundamental operations with polynomials: Practice must go beyond the immediate needs so that pupils may acquire confidence and facility.

(*3*) Factoring: the type $ax^2 + bx + c$.

(*4*) Radicals, the work being limited to using a table of roots of integers from 1 to 100, for instance, and making such changes as

$$\sqrt{240} = 4\sqrt{15}, \qquad \sqrt{\frac{5}{8}} = \frac{1}{4}\sqrt{10}, \qquad \frac{3}{\sqrt{5}} = \frac{3}{5}\sqrt{5}.$$

The arithmetic process of finding a square root can be reviewed Interpolation in tables should be studied. The Pythagorean relation should be assumed and used The formula for the area of a triangle in terms of its sides and other formulas involving square roots can be included

(*5*) Graphic representation: It is expected that less time will be spent on the construction and interpretation of statistical graphs than on the graphs of equations such as $y = ax + b$, $y = ax^2$, $xy = k$, $x^2 + y^2 = r^2$.

(*6*) Problems: It is expected that this course will include a wider range of algebraic problems than the course in general mathematics Problems about coins and ages, and other problems in which the answer must have been known before the problem was formulated are useful in developing skill in writing equations. Further, the restrictions stated above in (*1*) should not unduly restrict the material for problems or the methods by which a problem is solved, but it is important that the pupil should know that the purpose of such problems is merely to give practice and experience in the use of algebra. For example, the problem of finding three numbers such that one number exceeds twice the first by 5, and the third is 4 more than the first, the sum of the squares being 121, may be solved by writing

$$x^2 + (2x + 5)^2 + (x + 4)^2 = 121,$$

or by writing,

$$y = 2x + 5, \qquad z = x + 4, \qquad x^2 + y^2 + z^2 = 121$$

Although sets of three equations, one of which is a quadratic, are not mentioned in the equations in (*1*) above, nevertheless the pupil who has acquired the idea of substitution can solve this set, and can profit from a study of such sets In fact, the pupil should be encouraged to use this method since it is based on writing as an equation a given relation between quantities

(7) Formulas and generalizations: Since this course is for a special group of pupils, it is expected that much of the extremely simple work that has found its way into many courses can be omitted, such as finding the cost of some sugar at 6¢ a pound by using the graph of the formula $c = 6p$. The formula should be treated as a summary in a general form of a relation between variables. The significant problems should be studied in a general form (the so-called *literal* problems, mentioned on page 104), the formula being then regarded as a solution of a generalized problem.

(8) The trigonometric ratios: In case considerable time is necessary at the beginning of the course to correct for deficiencies in arithmetic, it may not be possible to give any instruction on the trigonometric ratios. Otherwise it should be included.

THE WORK OF THE TENTH GRADE

The work proposed for the tenth year consists chiefly of geometric material, and hence is best described as a course in geometry. It involves, however, more algebraic and trigonometric applications than has been customary to give in the traditional geometry course Accordingly, the year will carry forward the work in the various mathematical fields discussed in Chapter IV. If the work is suitably organized, it should be of high value to pupils whether they had general mathematics or algebra in the ninth year, and irrespective of whether they go to college. A reason for believing that the course may have considerable appeal comes from the fact that geometry touches human experience in a variety of ways, and in ways that can be made clear to young people—especially if the informal geometry of the earlier grades has sufficiently kindled the imagination. The pupil's experience with geometry may now be successfully rounded out by seeing the subject developed into a great logical system

What was said in Chapter V concerning the tenth year can serve as an approximate description of the content and general plan of the course. The chief change is that the course proposed here devotes less attention to arithmetic and algebra and correspondingly more time to geometry The lists of theorems accepted as satisfactory for the former course may be regarded as

merely minimum lists here. It seems reasonable to affirm that the more theorems a pupil actually has at his command, the more effective will be his work. Consequently the lists referred to should be supplemented when possible by other useful theorems. In particular, it is recommended that the course include work dealing with the length of a median; the length of an angle bisector in a triangle; the areas of inscribed and circumscribed triangles, squares, and hexagons, in terms of radii, apothems, or sides.

It may be mentioned that the theorems last mentioned and Heron's formula provide excellent material for mensuration problems employing algebra. In connection with the Pythagorean theorem, time may well be given to a review and extension of the work on radicals. Such changes as

$$\sqrt{\frac{1}{2}a^2} = \frac{1}{2}a\sqrt{2}, \qquad (a \geqq 0)$$

and the method of solving such equations as

$$x^2 + \left(\frac{1}{2}a\right)^2 = a^2, \qquad \text{and } 2x^2 = a^2$$

should be taught. The work on proportions need not be limited to the usual derivation of a proportion from similar triangles, but, in connection with a review and extension of the corresponding work of the ninth grade, can include inverse proportions, proportions considered as a special type of dependence, applications from science, and other useful problems. The work on similar triangles will give the pupils a better understanding of the trigonometric ratios; the problems need not be restricted to the very simple ones used in the ninth grade.

Although some of the mensurational aspects of solid geometry have always been treated in the lower grades, there is still disagreement as to how much solid geometry can be introduced into a course in plane geometry. Undoubtedly abler pupils profit by seeing some of the theorems of plane geometry extended through a consideration of related situations in space.

Three dimensional illustrations can be introduced safely whenever the pupil's understanding of plane figures is thereby deepened or supplemented. Such work cannot, of course, take the place of the thorough treatment that is demanded by engineering schools, and considering how little can be done, it would seem better in the course here contemplated to devote any available time to algebraic work, or to the use of non-mathematical material illustrating deductive thinking. The latter type of work adds to the appeal of geometry, and will undoubtedly increase in amount in the future.

THE WORK OF THE ELEVENTH GRADE

A systematic course in algebra is offered in the eleventh year for those pupils who had the course in general mathematics in the ninth year and also the course just described for the tenth year, and who continue their mathematical education. Since these pupils are definitely more mature mentally than ninth grade pupils and are somewhat experienced mathematically, the present course can be made more thorough, and can cover more topics than the algebra course previously described (which was planned for larger schools to offer in the ninth grade for certain pupils). It can, in fact, include essentially what is now commonly designated as *intermediate algebra*.

The course should consider more difficult cases and problems than are appropriate for the ninth grade course, and in addition to the topics in the outline on pages 107-109 it can include:

(*1*) Additional work on equations:
- (*a*) Theorems about the sum and the product of the roots of quadratic equations, and the condition for equal roots.
- (*b*) Extraneous roots of radical equations
- (*c*) Exponential equations, such as $3^x = 69$
- (*d*) Systems consisting of a linear and a quadratic equation, and simple cases of two quadratic equations
- (*e*) Graphs of quadratic equations.
- (*f*) Problems leading to equations of the types considered.

(2) Additional work on fundamental operations The work should contain more involved exercises than those in the ninth

grade, as, for example, operations with polynomials in which terms contain fractional and negative exponents.

(3) Factoring: the types $ax^2 + bx + c$, $ax + ay + bx + by$, $a^3 + b^3$, $a^3 - b^3$, and expressions reducible to these types, the factor theorem.

(4) Radicals; rationalizing denominators; geometrical material involving radicals

(5) Negative and fractional exponents; logarithms; applications involving more extensive computations than those of the ninth grade.

(6) Trigonometric work, including the use of four-place tables.

(7) Arithmetic and geometric progressions, with some applications to finance; the limit of the sum of a geometric progression when $r < 1$.

(8) The binomial theorem. (If this is included, some emphasis should be on the use of fractional exponents to derive approximation formulas. Finding the nth term, and related problems are of slight value to the pupil at this stage.)

No study of imaginary or complex numbers is suggested This implies the omission of all quadratic equations having complex roots

Students who had in the ninth year a course in algebra (such as is recommended as an option in larger schools) will now resume the study of this subject after having had the tenth grade work in geometry. It should be possible for them to cover in one semester the new topics listed above, with the inclusion of projects related to the main study[1] In the second semester of the year they can take one of the courses described below in connection with the work of the twelfth year

THE WORK OF THE TWELFTH GRADE

Under the plan of this chapter there is offered in the twelfth year a semester's work in each of the four subjects: trigonometry, solid geometry, social-economic arithmetic, and college algebra.

[1] In connection with the use of supplementary materials, a remark is in order It is undesirable to use titles that tend to magnify topics that may be presented in a very limited form For instance, it is an exaggeration to say that one is teaching statistics if he is merely having pupils draw distribution graphs and find medians Some instruction in central tendency and dispersion at least should be included before the title *statistics* is justified.

It is, of course, not expected that any one pupil will take all the courses, though, as indicated above, some pupils will have taken one of the courses in the eleventh year, and can accordingly cover two more of them in the twelfth year.

Two of the courses can be regarded as quite special, namely the course in college algebra and the course in solid geometry. As the name implies, college algebra is not a regular secondary study, but universities and colleges are glad to have students who intend to specialize either in mathematics or in one of the mathematical sciences ready to begin the study of the calculus when they enter college. They can do this if they have had a good course in trigonometry and sufficient algebra in high school, and the advantage that comes to them is above question. As to solid geometry, it is to be recalled that some instruction in mensuration is given in the earlier grades, while a development of elementary spatial insight was suggested as a part of the program for the tenth year. Accordingly, the thorough treatment of the subject that is described for the twelfth year may be desired only by pupils who expect to attend an engineering school that requires it for entrance, or who wish to specialize in mathematics, or who have felt a strong appeal in classic Euclidean geometry.

Both the course in social-economic arithmetic and the course in trigonometry should have a wider appeal than the two courses just discussed So far as the course in social-economic arithmetic is concerned, it needs merely be noted that it is especially designed to provide instruction recognized as important for the large group of secondary pupils who will not go to college. On the other hand, if some work with the trigonometric ratios has been given in the ninth grade, the pupil will already have acquired an appreciation of their importance, though he has had too small an amount of the subject to say that he has studied trigonometry. Since the trigonometric ratios hold the key to the solution of so many problems, trigonometry is a highly cultural study. A pupil may forget the details of computing inac-

cessible heights and lines, and of resolving forces and velocities into components, and he may lose his skill of combining several forces into one force or several velocities into one velocity; but if over a period of several weeks he has had extensive work with such problems, he will have a lasting and intelligent idea about one very fundamental kind of scientific work. Just as the theorem of sines, the theorem of cosines, and the theorem of tangents are essential for "practical" problems, so the formulas for the functions of the sum and the difference of two angles have a touch of the appeal of modern mathematics in its most aesthetic form. Finally, the almost endless identities that can be set for proof afford an example, perhaps unrivalled, of how rich may be the consequences of a small set of simple, wisely selected propositions. Every teacher of trigonometry has heard enthusiastic comments upon the rich significance of the subject, from pupils many of whom were not specializing in mathematics, and also from people whose school days are past.

Trigonometry. In the full semester that is contemplated for the work in trigonometry there should be time not only for many applied problems, but also for strengthening any neglected or weak parts of the pupil's earlier instruction in algebra. The work should be so well done that it will not have to be repeated in college. The suggested outline is.

(*1*) Problems requiring extensive calculations with four-place tables.

(*2*) Functions of any angle; functions of 30° and 45° and their multiples, the construction of angles when one function is given, the determination of the other ratios from one ratio, the reduction formulas; the graphs of the functions and their periodicity, the inverse functions.

(*3*) Functions of two or more angles.

(*4*) Properties of triangles, the law of sines, of cosines, and of tangents, the half-angle formulas, formulas for the radii of the inscribed and circumscribed circles, for altitudes, medians, angle-bisectors, and areas, Mollweide's equations and other checking formulas. The pupil should acquire considerable skill with the algebraic manipulations of the formulas and not merely a reading acquaintance with them.

(5) Identities and equations. This work is invaluable to the pupil who wishes to improve algebraic facility, and it should be spread throughout the course.

(6) The use of algebra and geometry in problems such as the following:

(a) Finding the functions of 22½°, 75°, 18°, and 36°, as a means of reviewing radicals and work on quadratic equations.

(b) Geometric proof of many of the trigonometric relations.

(c) The derivation of formulas which solve an entire class of problems instead of one specific problem.

(7) Radian measure with applications.

(8) Optional topics.

(a) Complex numbers.

(b) DeMoivre's theorem and the roots of unity.

(c) The components and resultants of vectors.

(d) Polar coordinates and the graphs of simple polar equations.

(e) Instruments used in surveying.

Solid Geometry. Attention has already been called to the fact that a semester course in solid geometry can be regarded as somewhat specialized, but the Commission would mention some significant and sometimes forgotten merits of the study.

(1) The traditional course in solid geometry known as Book VI not only contains certain theorems about lines and planes; it also offers an opportunity to enlarge the pupil's grasp of postulational thinking by a more thorough treatment of converses, inverses, and contrapositives of theorems (and the relations between theorems), of necessary and sufficient conditions (a topic that is seldom considered in plane geometry), and of indirect proofs; other topics of value are line-and-plane duality, and point-and-plane duality.

(2) There is opportunity for considerable algebraic work that may strengthen abilities the student will need later in parts of calculus. The theorems about frustums and spherical segments give a good opportunity for a careful review of radicals, rationalization of denominators, radical equations, and the manipulation of somewhat complicated algebraic expressions. For a review of literal equations there are exercises such as: "Express

the lateral area of a right circular cone as a function of the altitude and the radius of the base." "If the eight edges of a regular pyramid with a square base are all equal, express the length of an edge in terms of the total area"

(3) There is opportunity for computational work with use of logarithms. The problems should afford practice in organizing and presenting a somewhat long piece of work such as: "Find how many inches of rainfall are equivalent to the water from a hose through which the water flows for an hour at the rate of 10 gallons per minute, and spreads over a circle whose radius is 12 feet."

(4) In contrast with plane geometry, where the pupil constructs drawings with compasses and straightedge, the figures of solid geometry afford opportunity for the teaching of freehand sketching, which is so valuable to future architects and engineers, and is a worth-while accomplishment for other pupils. The straightedge may at times well be laid aside and the pupil be required to practice drawing as in the art class

(5) The work on loci can be very instructive, particularly the part that has to do with the use of coordinates.

(6) Although Cavalieri's Theorem and other means can be used to derive many of the formulas on mensuration which would otherwise involve a study of limits, some introduction to the notion of limits can at times be included. Such work will at least show the pupil the nature of certain difficulties in measurements and how the mathematician has met them

Social-Economic Mathematics. This course is a half-year study designed to acquaint the pupil with certain problems of modern society from a quantitative point of view. It assumes that the pupil has completed the work suggested for the ninth grade, but he may not have taken the work of the tenth or eleventh grade. It should be especially valuable to pupils in the commercial courses and to those who expect to study the social sciences later. Many of the topics are to a certain extent repetitions of similar work in the ninth grade, but it is expected that, because of the

lapse of time, the pupil will need a review. Moreover, on account of their greater maturity, pupils will be able to grasp more advanced concepts and also do more thorough work.

In the ninth grade many of the topics are presented largely from an informational point of view; here it is expected that more emphasis will be placed upon the quantitative aspects Since schools have only recently begun to experiment with this type of work, only suggestions of a quite tentative character can be made as to the possible contents of the course.

(*1*) Measurement and computations; degree of accuracy; approximate numbers; significant figures; short ways of multiplying and dividing, logarithms, computing machines.

(*2*) Such review of the fundamental operations with fractions, decimals, and per cents as may be necessary; practice in organizing and presenting problems that may involve only arithmetic, but that are more difficult than those of the ninth grade.

(*3*) The simpler ideas of statistical methods, considered early in the course so that the subsequent topics may be subjected to mathematical analysis as far as possible.

- (*a*) Construction of various types of graphs, including those with logarithmic (or ratio) scales on one axis.
- (*b*) Frequency tables; various types of averages and means; scatter diagrams.
- (*c*) Measures of central tendency and dispersion; correlation.

(*4*) Index numbers. Their construction and use in connection with commodity prices, real wages, cost of living, business cycles, etc

(*5*) Household budgets: the per cents spent on food, clothing, and shelter at various levels of income; cooperative enterprises.

(*6*) Installment buying: reasons for apparent high rates of interest; influence on business cycles; advantages and disadvantages.

(*7*) Investments: stocks, bonds, mortgages, investment trusts; banking procedures, periodic accumulations and payments; cost of home owning; annuities.

(*8*) Insurance: home, fire, theft, property, accident, etc.

(*9*) Taxation: property, sales, income, direct and indirect; the cost of government.

(*10*) Topics involving national policies, such as crop control, price fixing, social security, tariffs, foreign exchange, distribution of national income, etc

College Algebra. The work in college algebra, which is likely to be offered only by larger schools, assumes that the pupil has

had more than average success with his previous work on fundamental operations, exponents, logarithms, radicals, simultaneous quadratics, and progressions. To obtain the full benefit from it, the pupil should also be able to detect and remedy any deficiencies that may show themselves as his work progresses. For example, he should be able to grasp the definitions and operations with complex numbers without extensive help in the classroom. The work should be essentially the same as that in the course with the same title[2] taught in colleges during the freshman year, and should be so well done that the pupil can be excused from the corresponding course in college It is reasonable to suppose that this is possible, since high schools can give each week five recitations of forty minutes each for approximately eighteen weeks and the course comes when the work in algebra is still fresh in the pupil's mind. The course can include the following topics:

(*1*) Permutations and combinations; probability.

(*2*) Determinants (second and third order determinants evaluated by diagonals, higher order determinants evaluated by minors and by inversions), applications to equations

(*3*) Theory of equations: the fundamental theorem, equations reducible to quadratics; the factor theorem and the depression of equations; pairs of complex roots; relations between roots and coefficients, fractional roots, transformations of equations, the character of the roots; Descartes' rule of signs; location of roots by graphs, Horner's method of approximation

(*4*) Series· review of arithmetic and geometric progressions; harmonic progression, the binomial theorem, convergence and divergence.

(*5*) Mathematical induction

(*6*) Undetermined coefficients.

In the place of the college algebra described above, some schools may wish to offer a semester of mathematics not limited to algebra Such an alternative course would place less emphasis on some of the topics mentioned above, or not treat all of them,

[2] Since a large number of colleges and universities provide two different types of algebra courses for freshmen, the designation *college algebra* no longer means merely a course in algebra that is taught in college, but a particular course

and it would involve an introduction to the elements of analytics, differential calculus, the mathematics of finance, and parts of elementary statistics. Cultural courses of this type can be made very valuable to all pupils, and, since survey courses are beginning to appear in college, such a course might excuse a student from a required college course, provided he did not intend to pursue a systematic study of mathematics.

CHAPTER VII

THE PROBLEMS OF RETARDATION AND ACCELERATION

"Undoubtedly philosophers are in the right when they tell us, that nothing is great or little otherwise than by comparison.

—JONATHAN SWIFT, GULLIVER'S TRAVELS

VARYING CAPACITIES OF PEOPLE

IT MUST SEEM incredible to many teachers that the statement in the Declaration of Independence relative to equality should ever have been given a meaning other than a political one, for classroom experience shows that the abilities of pupils are far from uniform Jefferson, who wrote the celebrated words, and who was a sincere friend of popular education, was quite pessimistic concerning benefits of general higher education; his scheme of education was in fact definitely selective In the Constitution the postulate of the Declaration finds no place, and from the earliest days of the Republic schoolmasters and school teachers have striven with the problem of the slow pupil and the bright one. The dunce cap is not just a fiction of the present day, for it actually was used both to frighten the able but indifferent child into activity, and to punish, unjustly, the truly backward pupil. Fortunately, the picture of the past is not all of so somber a nature, and it is made more agreeable by the recollection that scholarships for bright pupils are also an old institution. Within recent years, indiscriminate assumptions as to native ability and intelligence have been sharply called into question by a number of psychologists, notably those of the behaviorist school. The doctrines they pronounce help to

keep a general balance in the problem of teaching, even in the mind of a teacher who does not accept them but holds that the effects of environment must be subject to distinct limits. In the present chapter the hypothesis of a wide range of abilities or capacities is frankly accepted, and the bearing it has upon mathematics instruction is examined.

Although it is easy to be aware of differences of intelligence, it is quite another matter to measure them—and it is still another question to devise proper administrative and instructional procedures to take account of them. It is only within recent years that tests have been constructed to measure intelligence. Sir Francis Galton is usually credited with the initial attempts to measure intelligence because of the work he did in England about 1885. When Binet was given the task of assigning children to a home for the feeble-minded in France, he found that these children were unable to answer certain series of questions that could be answered by normal children of their ages. In this way he established tests for various age levels, and he was led to measure intelligence by comparing the mental age of a child, as revealed by the tests, with his chronological age. Later Terman greatly extended and refined the procedure of Binet, and in 1916 published the Stanford-Binet test, a revision of which was made in 1937. Group intelligence tests, first devised by Otis, are an important by-product of the World War, and they are more widely used throughout the schools in the United States than any other tests.

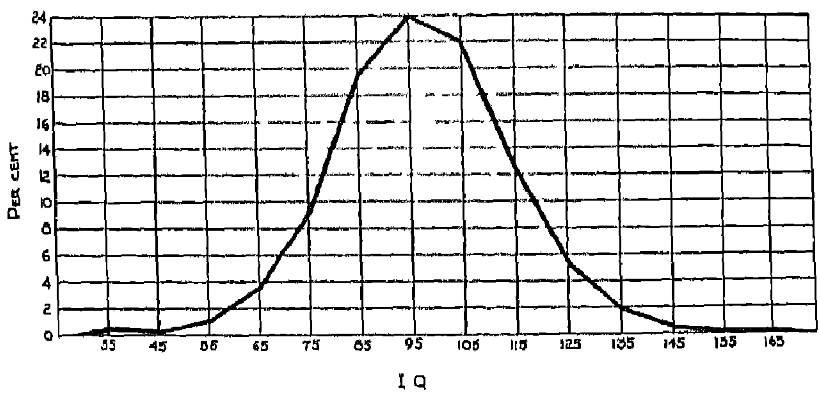
FUNCTION OF INTELLIGENCE TESTS

A good intelligence test is probably the best single instrument at present known for measuring general scholastic aptitude. Nevertheless, some teachers who do not understand the function of such tests and who have seen them used incorrectly, appear reluctant to avail themselves of this contribution of psychology.

In some schools the tests have been carelessly administered

and badly scored, and the results have been unreliable Extravagant claims that have been made for the tests by over-enthusiastic novices have also weakened their prestige. Terman and Merrill in *Measuring Intelligence* describe the function of the tests as follows.[1]

> The expression of test results in terms of age norms is simple and unambiguous, resting upon no statistical assumptions A test so scaled does not pretend to measure intelligence as linear distance is measured by equal units of a foot-rule, but tells us merely that the ability of a given subject corresponds to the average ability of children of such and such an age. This was all that Binet claimed to accomplish, and one can well doubt whether the voluminous output of psychometric literature since his day has enabled us to accomplish more



From Terman and Merrill, Measuring Intelligence, Fig 1, p 37, Houghton Mifflin Co, 1937

Distributions of Composite L–M IQ's of Standardization Group
Ages 2 to 18 $N = 2904$

As a result of his first investigations, which are generally well known, Terman divided school children into the following three divisions.[2] (*a*) superior group, 20% of school population, IQ above 110, (*b*) average group, 60% of school population, IQ from 90 to 110, (*c*) slow group, 20% of school population, IQ

[1] Terman, Lewis M and Merrill, Maud A *Measuring Intelligence*, p. 25. Houghton Mifflin Co, Boston, 1937

[2] Terman, Lewis M *The Measurement of Intelligence*, pp 78-79 Houghton Mifflin Co, Boston, 1916.

below 90 These early findings have been confirmed by many later studies, including others by Terman, wherein he shows that the intellectual differences of American-born white children of ages 2 to 18 can be pictured as in the graph on the opposite page [3]

RECENT CHANGES IN HIGH SCHOOL POPULATION

The great growth of the enrollment in the secondary schools of the country has been noted in Chapter I, where some comments were made as to its cause. The increase in numbers has resulted in a great change in the type of pupils who are admitted to the high schools While formerly they came chiefly from the upper economic level of society, they are now recruited from practically all levels. It is not true, however, that the high school is less of a preparatory school to any significant degree than it was at the beginning of the century, for university and college enrollments have practically kept pace with high school enrollments [4] This fact, interesting in itself and often overlooked, does not mean, however, that the high school problem has not significantly changed, for present data show that it is not only the children of superior or average intelligence who are remaining in school beyond the elementary grades. The opposite view is in fact indicated by such data as the following, which show the approximate intelligence distribution of 2,241 children who entered the eighth grade in Rochester, New York, September, 1938:

[3] The graph is based on data in Terman and Merrill, *op cit*, p 37. It gives the distribution of the composite scores that show the IQ's obtained from two forms of the revised Terman scales administered to a standardization group of 2,904 children of from 2 to 18 years of age, expressed in terms of per cent of the cases, at ten point intervals

[4] Many statistics bearing on the question can be found in the reports of the U. S. Office of Education; but it is difficult to tell just what figures one should select, on account of differences in classification, etc. It appears, however, that both college enrollment (broadly interpreted) and high school enrollment are ten to twelve times what they were in 1900 During the period of 1900 to 1935 the high school population was about five times the college population, and the ratio is increasing somewhat

(*1*) Superior Group	17% of 8th grade	IQ 110 or above
(*2*) Average Group	59% of 8th grade	IQ from 90 to 109
(*3*) Slow Group	25% of 8th grade	IQ below 90

There is close agreement between these figures and those of the original Terman study, which was based upon unselected groups.

In the mathematics classroom the extremes in ability of pupils stand out clearly. Probably several factors are responsible, but whatever may be the explanation it is certain that mathematics teachers face a serious problem if they are to make the subject both cultural and useful to the unselected mass of secondary pupils who are now in our schools.

In the immediately following sections of this chapter a discussion will be given of general methods and devices that have been employed to take account of the wide range of pupil ability; in later sections the question of adapting them to the mathematics program is considered. Perhaps no other part of the Report deals with a problem at the same time more important and more baffling. Many teachers who feel themselves competent to deal with pupils of normal ability are perplexed, if not in despair, over the question of instructing those of inferior ability, while others feel that they are not providing adequately for superior pupils. Experiments are being carried on in different school systems, but most programs of instruction are still so experimental that few results have been published Considerable difference of opinion exists even on basic matters, and much work remains to be done before some of the uncertainties can be removed. This perhaps is a field in which the personality and resourcefulness of the individual teacher will always play a very important role Few clear and trustworthy sign-posts exist in this area of the educational problem

PRESENT GRADE GROUPINGS

The organization of schools into grades is for the purpose of facilitating instruction, for system and arrangement are essen-

tial in any school, especially in a school that deals with large numbers The old custom of attempting to retain a pupil in a grade until he has successfully passed the work assigned to the grade is certain to bring together in the upper grades of the elementary school pupils of a wide variety of ages. Those who barely pass the first grades are likely to fail soon, and their sojourns in different grades will be prolonged as they proceed. Just how great an assortment of ages would result if rigid standards were enforced has been shown by Coxe. Taking data from many sources and using batteries of tests, he has shown by an illuminating and unusual graph how great is the variation of achievement in children of different chronological ages and at different grade levels.[5] One illustration will be given of the information that can be read from the graph of Coxe. The work that is regarded as normal for the sixth grade can be done by the upper quarter of all children at ages between nine years and ten years six months. The next fifty per cent of the pupils, who may in general be regarded as normal, reach the same degree of achievement at ages ranging from ten years six months to fifteen years. The remaining quarter of the entire group never actually reach the sixth grade standard. A reluctance to have in the same grade children of different ages has caused some administrators to abandon almost completely the policy of requiring that a grade stand uncompromisingly for a reasonable but definite achievement. The resulting practice of passing pupils whether they do good work or do poor work raises serious questions. In the minds of many persons, including some pupils themselves, it seems dishonest to pass to a new grade a pupil who has plainly not done the work of the last grade, though he might readily have done so.[6] It is clearly not within the province of this Re-

[5] Coxe, Warren W *Our Educational Problem as Revealed by Pupil Variability* A mimeographed publication of the Division of Research, State Educational Department, Albany, N Y, 1937.

[6] A strong attack upon the practice has been made by John L. Tildsley, in *The Mounting Waste of the American Secondary School.* Harvard University Press, 1936.

port to debate the merits or demerits of the new policy; but it is to be noted that when it is carried out there is a wide spread of achievement in the upper grades, with increasing difficulty of instruction

ABILITY GROUPING

It is difficult to say just when and where efforts were first made to achieve homogeneous ability grouping in different grades; but in 1920 Detroit, among the large cities, became prominent for dividing pupils into superior, normal, and slow groups, the groups being called the X, Y, and Z groups respectively The practice and designations have been taken up in other school systems; in some cities separate schools have been established for slower pupils. Since slower pupils give rise to serious problems on account of failure, it is natural that more solicitude should be manifested for them than for superior pupils who, because they succeed well with their work, cause no comparable problem for a school Recently, however, there has been a tendency to emphasize the importance of making really adequate provision for the most able pupils Since it is from this class that leadership in all phases of life must come, it is a responsibility of the school to see that its pupils have their powers fully developed and their interests fully awakened in a wide variety of activities. The size of the school determines the school years in which the formation of such groups as the X, Y, Z groups can be carried on with success; in some systems it has been carried through all the years of the high schools in certain subjects

Homogeneous ability grouping has not always led to satisfactory results. Some of the failures may have been due to inadequate recognition by administrators and teachers of the fact that extensive adaptations of instruction are necessary to insure success In some instances the prescribed course of study, the character of the standards, the subject matter chosen, as well as the methods of teaching, have been too nearly the same

for all pupils. Since ability grouping allows a teacher to work with a homogeneous group, it is possible to adapt instruction to a class as a whole. No competent teacher will complain of a class, all members of which are able and industrious If the material taught to a group of slow pupils is wisely selected, and if due attention is paid to ways of arousing their interest, and to methods of instruction, satisfactory results should be possible with such pupils. The belief that it is well to have a few bright pupils in every class because they will serve as a spur to the others has been overworked as an argument against ability grouping; it is in fact a spurious argument. The pupil who can inspire others effectively is one whom they recognize as being somewhat comparable to themselves and who is obviously doing well. In order to raise the level of general achievement we should have in our classes a few pupils who are industrious and interested in their work, but whose abilities are comparable to the abilities of the other members: it is neither a genius nor even a brilliant student that is required in every class. When groups are approximately homogeneous, it is possible to adapt subject matter, methods of teaching, and standards of achievement to the needs and capacities of the individual pupils When superior pupils are segregated, the teacher can devote the entire class period to developing their abilities unhampered by the time-demanding errors of the dull. Segregation, wisely administered, gives to the slow pupil a chance to distinguish himself in his own group, and this is beneficial to his mental health.

After considering very carefully the data that now exist with regard to ability grouping, Coxe wrote as follows:[7]

> The contribution of research studies tends to give some evidence that, at least in traditional subject matter, pupils make more development under a system of ability grouping than under other systems of grouping. . . .

[7] Coxe, Warren W "The Grouping of Pupils" *Thirty-fifth Yearbook of the National Society for the Study of Education,* Part I, p 309. Bloomington, Illinois, Public School Publishing Co., 1936

> The trend of the argument in this yearbook . . is that much higher achievement may be expected when pupils are properly grouped. The highest intellectual developments can take place only when pupils are placed in such situations as will challenge their full capacity.

This is a temperate statement, but it is a better endorsement of ability grouping than an over-enthusiastic eulogy would be In the lines quoted it is not stated that abilities of a group should be somewhat homogeneous, but the context leaves no doubt on that point. On the question of challenging the full capacity of pupils, which is emphasized by Coxe, one can return to the thought expressed above, that another person can stimulate us effectively if we know we can do as well as he, provided we make full application of our powers.

When funds permit, it seems to be the tendency in the lower grades to enroll 25 pupils in slow classes, 35 in average classes, and 40 in bright classes; in the high school the average class sizes are smaller. Some administrators, however, make the group of average pupils larger than the group of bright pupils, which tends then to be still more selective [8]

It has been found that a cumulative record of a pupil, with entries supplied by his successive teachers and accompanying his progress through school, furnishes data that are invaluable in assigning him to the section in which he will do his best work. Since pupils differ not only in abilities but also in experience, industry, interests, health, and amount of knowledge accumulated (which may not be proportional to mental capacity), and since all these factors influence his future success, various types of information should be entered on his record. In addition to intelligence rating, the record should give such other facts as actual results obtained in achievement tests, especially in mathematics and reading, teachers' judgments as expressed in school marks, age, physical defects and general health, special abilities and interests, vocational plans, eco-

[8] Coxe, *op. cit*, p. 313

nomic status and occupations of the parents, and language spoken at home.

Adjustment of standards of performance to the abilities of the different sections raises both educational and psychological questions. In some schools achievement marks over the entire marking range, for example, A, B, C, D, E, F, Inc., are awarded to pupils in all the groups, marks being then frankly relative rather than supposedly absolute in character Under such an arrangement a pupil in the Z group may obtain the highest rating and one in the X group the lowest. It is of course absolutely essential that the school record show the type of section in which a pupil was enrolled, and it is equally important that the matter be made clear in transcripts to colleges or in reports to possible employers

What has been said assumes that sectioning has been a recognized practice of the school, systematically provided for by the administration. When this is not the case a close approximation to it can be carried on by teachers themselves in case there are two or more sections in the same subject. Thus if two sections in mathematics meet at the same hours as two sections in some other study, a division along ability lines can be carried out, though of course compromises will have to be made in the placement of some individuals. When this procedure is resorted to it may be that the school records will not show the type of section in which a pupil has done his work, and the perplexing problem of grades for differentiated sections will arise. Efforts may be necessary to prevent an able pupil from going into the slower section because of the ease with which a high grade can be won in it.

It is to be remarked that some school systems have abandoned ability grouping, after having experimented with it Sometimes it is stated that the procedure is undemocratic, and that a social or psychological stigma attaches to membership in the slowest group. The Commission does not believe such reasons to be sound, though it clearly recognizes the difficulty of plac-

ing pupils properly and is aware of difficulties of administering the program successfully.

OTHER METHODS OF CARING FOR ABILITY DIFFERENCES

When the number of pupils in a grade or the enrollment in a subject is too small, it is not possible to carry out the segregations previously discussed. Nevertheless there are procedures by which differences in ability can be allowed for, either by teachers in the method of conducting classes, or by the school through the provision of what is sometimes called an "oppor tunity room."

Where all the work of a pupil is with the same teacher, it is possible to approximate ability grouping in some studies at least. Appropriate divisions can be carried out in grades seven and eight if the pupils in these grades do not have different teachers for different subjects. It is probable, however, that two rather than three groups should be provided for in any one study.

When pupils reach the place where they go from room to room for instruction under different teachers, differentiated assignments can still be carried on, but difficulties may arise unless the planning is carefully done and vigilance is exercised. One procedure is to have pupils study the same general topics, with assignments adjusted to their varying alertness and powers. Perhaps the most feasible method of instruction is to have a common element for all pupils in a day's work, with extra assignments for the abler group. The extra work given these pupils should not be merely more of the same material given the rest of the class, unless it calls for a distinctly higher order of understanding; the general aim should be to develop a greater breadth as well as more complete mastery. The extra work must of course be checked by the teacher and carefully appraised, and assignments must be handed in frequently. Regular days may be set for the two sections of the class to meet separately (in the same room if necessary), the abler group for

discussion of their special material, the other group for further consideration of what is required of them, and for reviews. At such times written work can be provided for the group that is not having discussion. Efforts should also be made to have the stronger group carry on work in special meetings by themselves

Some teachers report that they have been successful in dividing classes into as many as three groups, using different texts with the different groups, quite as if there had been a sectioning into distinct classes Undoubtedly a very competent teacher would be required to conduct a class in such a fashion, and the procedure tends to put considerable responsibility upon the pupils. No one will deny that at least on the surface teaching reaches its ideal when the teacher adopts the general practice of helping pupils to instruct themselves, the question is one of the character of the results obtained Methods of "progressive education" call for such a teaching procedure, and differentiated assignments when fully individualized proceed in much the same way, with pupils having a voice in deciding the standards they wish to attain. Under such circumstances a class meeting is often quite unconventional, and to many teachers it sometimes appears disorganized. To what extent highly individualized methods of instruction can be used profitably in public secondary education will probably long be a subject of debate Success will depend not only upon the teacher but upon the size of the class and the personnel that comprise it Neither a general endorsement nor a broad disapproval would seem to be justified

In the opportunity room, or the adjustment room as it is also called, pupils are given a chance to make up deficiencies in a specific subject, such as mathematics. A good teacher, relieved of some regular instructional duties, should be in charge of the room. Pupils are assigned to the room by teachers of regular classes for short periods of remedial work. Such treatment is especially useful for pupils who have been absent, for transfers from other schools using different courses of study, and for

pupils suffering from special disabilities. Since the work is largely individual a teacher may diagnose difficulties quite accurately and apply proper remedial treatment; on the whole the plan is a better one for remedial work than for presenting new material to a slow pupil, as there is not the benefit of a class recitation The plan can become somewhat expensive; but it can be the means of preventing many failures, and failures are still more expensive.

Provision of special instruction for superior pupils by a plan similar to the opportunity room as here discussed will be mentioned later in the chapter.

CHARACTERISTICS OF BACKWARD CHILDREN

Pupils who are designated as "dull normal" usually have an intelligence quotient between 70 and 90 In this section attention will be limited to this group, for the defective children, whose intelligence quotients are below 70, are not often found in the mathematics classes of the secondary school, and they constitute a special problem into which this report cannot enter An explanation should be made concerning the reason for using the intelligence quotient (IQ) rather than the educational quotient (EQ), a measure found helpful in certain connections. Although there is a high correlation between the two measures, the group of pupils with low EQ's includes bright and average children who merely will not work, and some who, because of physical reasons, are unable to do so. Since such bright children may be present in considerable numbers and since they have great possibilities of success when they exert themselves or have their disabilities removed, general conclusions based upon children with low EQ's may be erroneous For this reason, it is the child with the low IQ and not the child with low EQ who will be considered as backward in the present discussion

In order to deal successfully with backward children a teacher must first of all know some of their traits. From an extensive

study Schorling has formulated the following general conclusions about the dull normal pupil:[9]

1. The dull differ from the normal not in kind but in degree.

2. As regards sensory and motor capacities the dull are not far from normal

3. With respect to instincts and emotions also the dull approach much nearer the norms than they do in intellectual traits.

4. The higher mental processes differentiate the dull from the normal and the bright.

This lack of system in the minds of the dull greatly limits the amount of transfer of training.

It is very difficult for dull pupils to detect an absurdity in an illogical statement.

The dull pupil is extremely weak in forming associations between words and ideas.

Dull pupils are limited with respect to imagination.

The dull have an inadequate memory.

The association of factors, an important element in recall, is weak.

Most important of all, the dull pupil has difficulty in generalizing.

Then, too, the mentally limited pupil is weak in evaluating his efforts.

5 Dull pupils have a short attention span

6 The problem of the dull normal is fundamentally one of mental health.

7. The dull pupil's responses are less reliable than those of the normal pupil.

8. Experiments as regards relative amounts of play are as yet meagre but seem to indicate that the dull children play less than the normal

9. The dull normal usually responds well to responsibility for little extra jobs than can be delegated to him.

10. Though the difference between teachers is greater than the difference between school subjects, in traditional secondary schools Latin and algebra hold the unenviable distinction of being the most difficult subjects for the dull normal pupils.

11 The curve of growth toward a specific maturation under constant environmental influences appears to be the same for the dull as for the bright.

[9] Schorling, Raleigh. *The Technique of Instruction for Dull Normal Pupils*, pp. 15 35 Bureau of Educational Reference and Research, Ann Arbor, Michigan, 1934.

Not all backward children will possess all these deficiencies in equal degrees; but no teacher can read the list without having his desire strengthened to reach this group of pupils more effectively. An awareness of such specific traits as those given above should make it possible to select appropriate material and carry on instruction far more successfully.

Other investigators who have worked on the problem have published important findings. Thus from a study of the growth curves for a number of children in learning the tasks in algebra as measured by the Butler and Breslich tests, Beck concludes,[10] "The data support the fact that the slow group grows in the same proportion as the fast group though on lower levels of development." If the aim of teaching is to help pupils improve and to aid them in reaching the state of understanding and achievement of which they are capable, such a pronouncement as this should not go unheeded.

There is another very important aspect of the problem that must be considered before a decision is made as to the programs suitable for backward pupils. Burt, the English psychologist, in an exhaustive study of the slow child[11] compared 400 backward boys and girls each with a normal child of the same age and sex attending the same school. He discovered that the average intelligence quotient of the backward children was approximately 78. But physical comparisons were equally striking. The backward children were on an average nearly one inch shorter In weight, both the defective and the backward were about three pounds below the standards. And, contrary to popular belief, he found that slow pupils do not excel in manual skill as a compensation for mental ability, but in this as in academic work, they rank below the normal and the defective.

[10] Beck, Hildegarde. *An Analysis of the Relative Achievement in Mathematics of Slow and Fast Groups in Junior High School Level*, p 48. Unpublished Master's thesis, University of Michigan, 1933

[11] Burt, Cyril, *The Backward Child*, pp. 155, 156, 267, 269 D Appleton Century Co., New York, 1937

CHARACTERISTICS OF BRIGHT CHILDREN

Baker has observed[12] that although the superior pupil may be similar to the backward child in responses to very simple situations, the former has the ability to solve complex problems by creating and manipulating many associations, a thing the other child cannot do. Bright pupils enjoy abstractions, as a general rule, they are capable of reasoning, they have initiative, imagination, associative memory resourcefulness, and they read with understanding. But much more than this can be said of them: they wish to make discoveries for themselves, though after the discoveries are made they may be bored by the practice that must follow if they are to retain what they have achieved. Because of the dislike that bright children have for practice, some teachers consider it unnecessary for them, and omit it from their programs. The omission of such an important aid to mastery is likely to be detrimental even to bright children, for it may lead to failure or near failure, or at least to a lessened efficiency.

It may be a surprise to many persons to know that statistics show that, considering their mental ages, it is the superior pupils and not the slow ones who often form the retarded group in our schools. The competition of other activities, the ease with which they excel in a heterogeneous group, the lack of challenging work suited to their capacities and interests, the agreement sometimes existing among pupils that a "gentleman's grade" is one given for low achievement- all these factors have resulted in producing a deplorable retardation on the part of some of the ablest children. Such a condition should be corrected; for since it is from the abler pupils that leaders should come, their years in school should be spent in serious preparation for the work that they will later do.

In his detailed study of the traits of the bright child, Terman

[12] Baker, Harry *Characteristic Differences in Bright and Dull Pupils*. Public School Publishing Co., Bloomington, Ill., 1927

has found[13] that the gifted excel not only in intellectual traits, such as originality, will power, capacity to persevere, sense of humor, and common sense, but also in physical growth and in general health, a result in harmony with that of Burt, which was mentioned above.

TEACHERS FOR BACKWARD AND SUPERIOR GROUPS

When an actual segregation into X, Y, and Z groups can be made there arises the important question of the selection of the respective teachers.

There is a growing conviction that for the two extreme groups, for the slow Z group quite as well as for the fast X group, especially good teachers—though perhaps with different traits—are essential. This has not always been the case, for in times past teaching dull pupils was thought to mean little more than entertaining them until they reached an age at which they could legally leave school. This mistaken opinion often resulted in assigning slow classes to the least successful or the least experienced teacher, or to a teacher who would not protest too much. The recent greater concern for the slow group of pupils comes first from the knowledge that many pupils in that group can do more than was formerly realized, and second from a stronger impulse to help them to the full development of which they are capable. Thus to an increasing extent slow classes are assigned to very skillful members of the teaching staff Under the teaching of such competent instructors backward pupils may do much, while they would fail to advance significantly under poor teaching, or actually develop undesirable habits of behavior.

A chief requisite of a teacher of slow pupils is a firm belief in the worth-whileness of the enterprise and a conviction of the possibility of success: no teacher who thinks that the task is futile should be so employed. Faith in the value of the work

[13] Terman, Lewis M. *Mental and Physical Traits of a Thousand Gifted Children*, p 634. Stanford University Press, Stanford, Calif., 1926.

will come from a study of the capacities of backward pupils, for such study gives a knowledge of what they can do and of what is beyond them; it also shows the way to awaken their interest and keep them at their tasks In the teacher there must be sympathy, there must be patience, and there must be the imagination that will lead to the presentation of a topic not just once and in one way—a thing the beginning teacher often regards as sufficient—but repeatedly and from different viewpoints.

In order that their full capacities may be developed, superior pupils should have superior teachers. The teacher should be a distinctly intelligent person, and should possess not only good social qualities but discrimination Though such a teacher should manifest an awareness of current problems and take an interest in them, he should also reveal a firm attachment to those great underlying achievements and interests that gave dignity and nobility to human life He should have a broad range of information, and an extensive knowledge of the special subject taught and its relation to other fields

The problem of controlling conduct cannot be ignored Though the backward child may desire to attract attention to himself, his means of doing so are somewhat restricted Since he cannot arouse attention through really superior achievement —as a superior child can do—he may resort to some crude form of exhibition. In order to be full master of the situation, the teacher needs not only firmness but tact, and should seek to make the simple tasks that are set for the pupils as satisfying and as enticing as possible The occasional problems of discipline which arise with both superior and dull children should be seen to call for direction and guidance rather than domination, the aim being to develop intelligent self discipline in an atmosphere of mutual respect for character and worthy achievement. Some of the elements necessary for teachers of backward and superior pupils seem to be natural inherent traits; but study and training are very important in developing these native

abilities. The increased attention that the problem has recently received in teacher training programs is an encouraging sign.

TEACHING PROCEDURE

Since it is now well known that children may be retarded in school because of physical as well as mental disabilities, backward pupils should have physical as well as mental examinations, and the results of the medical findings should be followed up. Children who have been classed as failures are often able to make average progress by the correction of physical defects, for instance by fitting glasses, providing adequate diet, transferring to open air school, or removing diseased tonsils or adenoids. But once it is ascertained that unsatisfactory progress is due to actual lack of mental ability, special teaching methods as well as special subject matter are indicated, and should be employed in accordance with the special plan that the school is able to adopt.

Burt[14] gives the following advice to the teacher of the backward pupil:

> The whole type of instruction needs to be radically changed. . . Perhaps the most difficult point to bring home is that there is no single method appropriate to the backward child as such. The essential need is a teacher with an experimental outlook and adaptable turn of mind. What particular changes should be made will depend in every instance on the underlying causes of the trouble: hence the main key to success is to vary and modify the teaching until it fits the individual child.

Slow pupils, like very young children, learn best through experiences with concrete things. They must handle, measure, count, draw, make models, construct graphs, go on expeditions, hear talks, see pictures of things, etc., until the quantitative or spatial characteristics or relationships to be taught are really understood. After that the teacher may lead very gradually to the abstract, returning frequently to the concrete to fix and illustrate generalizations. Slow children can seldom make generalizations or discoveries, except of the simpler kind. They profit

[14] Burt, Cyril, *op. cit.*, p. 116.

from detailed explanations and enjoy repetition of familiar material. The teacher must discover all the short steps by which their slow minds must travel, and be ready to aid with all the steps in sequence. Much of the natural vocabulary of the teacher must be put aside, in order to talk in the everyday language of the pupils who are being taught

Reading difficulties form a special weakness to be corrected, and the teacher of the slow-learning will profit from a careful study of books on the teaching of reading. If children can be taught to read successfully, they can often do mathematics which they formerly did not understand merely because they did not comprehend the meaning of the words in which the mathematical concepts were explained. The matter of reading and vocabulary, so necessary a consideration in all backward pupils, is still more important in the case of those who do not speak English in their homes.

Oral reading of mathematical material, which may well be motivated by casting it in the style of a play, affords an excellent point of departure not only for a discussion of the textbook but also for the comprehension of the meaning of non-technical words and mathematical terms. Newspaper clippings furnish a continuous source for reading and discussion of quantitative expressions, both technical and non technical, as, for example, the association of a tube of any kind with a cylinder.

The span of attention of the slow pupil is short, and this fact makes it essential that there should be variety within one period and that the units of work should be short. Although abstractions present great difficulties to the backward, slow pupils should be allowed to reason and discover in the realm of simple ideas as much as they are capable of understanding. When they are in a homogeneous group they often enjoy doing this. Since it is with concrete material that they will be most successful, the classrooms of slow pupils should be especially equipped with rulers, tapes, protractors, compasses, squared paper, models, advertisements, and business forms. Slower pupils actually show

ability in performing easy computations and in applying simple rules that may be learned by rote. Their range of ability in arithmetical computation includes work with decimal fractions and common fractions as well as whole numbers [15]

It is not necessary to devote as much space to the discussion of the traits of superior pupils as was given to the traits of slow pupils. This statement is not meant to imply that methods and means of dealing with such pupils are always well understood, but, since superior pupils should be instructed by a superior teacher, there should be in this type of instruction a meeting of minds that have important qualities in common and a contact of personalities with congenial impulses and tastes. Among the pertinent characteristics of superior children may be mentioned: capacity of sustained attention with a consequent ability to work with assignments covering several days, the power of dealing successfully with abstract ideas, and a readiness in understanding applications of an advanced or difficult type.

A further word with regard to the function of practice seems appropriate. Even though the superior pupil may be somewhat rebellious with drill, and may be attracted more to ideas than to manipulation, extensive practice is often necessary It is one thing to understand a process when it is explained, but quite another to absorb and make it a part of one's working equipment, ready for immediate use. Superior pupils should be able to grasp the truth of this fact, a skillful teacher of course helps relieve monotony by introducing variety Although the backward pupil may not grow restless with repeated drill, and may even take satisfaction in a sense of achievement, the teacher must make sure that the pupil understands the reasons for what he is doing. The teacher must also make sure that the pupil is forming habits of correct response and is not drilling upon errors, for the backward pupil is quite uncritical of his own work.

[15] Potter, Mary A *A Study of the Errors Made in Computational Arithmetic by Children with High and Low Intelligence* Unpublished Master's Thesis, University of Wisconsin, 1930

Reviews must be frequent, and there must be considerable reteaching of backward pupils. To the slow child merited praise and satisfaction in previous accomplishments are effective stimuli for new attainments. In this connection Burt observes:[16]

> It is amazing to see what a patient teacher, who is prepared to make full allowance for such temperamental creatures and to plan his syllabus along appropriate lines, can extract from this unpromising material.

What has been said above applies most specifically to the teaching of the Z and X groups when X, Y, Z grouping is employed. When that grouping is not possible, and a plan of differentiated assignments is used, the teaching problem is quite different. One teacher must then deal with the whole range of abilities, but a knowledge of the traits of abler pupils will help in the problem of special assignments for them, and the reviews suggested for the remaining pupils will give opportunity to deal appropriately with the slower members of the class.

THE MATHEMATICS FOR BACKWARD PUPILS[17]

The Commission has already expressed the view that mathematics should be required through grade nine. It believes that this amount of mathematical study is desirable not only because it is useful but because it helps in a unique way toward intelligent adjustment in the present-day world. The programs outlined in the two preceding chapters had in mind average pupils for the lower grades (seven, eight, and nine), and able, though not necessarily superior, pupils for the upper grades. It is necessary to consider now how the work suggested for grades seven, eight, and nine can be modified for backward pupils.

Although a great deal of further experimentation is required to check results, the Commission urges that mathematics should

[16] Burt, Cyril, *op. cit.*, pp. 512-549.

[17] Throughout this section a backward pupil is regarded as one who has difficulty with all his studies. Some of the remarks made also apply to the pupil for whom mathematics is a study of special difficulty and who may appropriately be in a mathematics class with the general slow-moving pupils.

not be regarded solely as a study of utility in the case of backward pupils, with emphasis exclusively on problems they may actually need to solve. It believes that for backward pupils as well as for able pupils mathematics should be in part cultural and informational. The future life of a backward pupil is destined to be quite circumscribed intellectually, and even a limited background of appreciations helps make him a better citizen. Such an aim seems practicable if proper material of instruction is chosen and if there is good teaching.

Backward pupils should not be taught only arithmetic, through the mistaken belief that arithmetic is the simplest mathematics since it is the most elementary. Parts of algebra and geometry are simpler than parts of arithmetic; furthermore, they may be more useful and more broadening. Slow pupils should be taught the simple parts of different divisions of mathematics.[18] In all the branches of mathematics that are taught there must be a preponderance of concrete material and experience, abstract parts being introduced slowly and with moderation. A variety of subject matter can be successfully taught if it is presented in concrete form, if it is properly motivated, and if time is given for its mastery. Without too great an expenditure of time the slower pupil can be taught the use of letters for numbers, evaluation of simple formulas, work with ratio and proportion, the solution of easy problems, and the reading and construction of various types of graphs.

In geometry the slow pupil can at least make simple drawings and constructions; he can recognize and enjoy geometric forms in nature, art, and architecture; he can do considerable work

[18] Mallory reports that, in the country as a whole, if mathematics is taught to slow children in the ninth grade, it takes one of the forms. (1) rather traditional algebra with lowered standards, (2) modified algebra, (3) review of arithmetic or business arithmetic; (4) shop mathematics; (5) some type of general mathematics. His own recommendation is that "the material taught should be so called general mathematics including social uses of arithmetic, practice in computation, simple algebra, geometry, and numerical trigonometry." See Mallory, Virgil, "Mathematics for the Slow Pupil" *The Mathematics Teacher,* November, 1933, pp 391 398.

with mensuration, and he can make scale drawings. In some instances a little work in simple geometric demonstration can be done. Elementary work with trigonometric ratios can be done by many backward pupils if enough time is given to it.

As a matter of significant information that will help form an intelligent outlook on life, slow pupils can appreciate something of the role of mathematics in civilization through the story of numbers and the history of measurement. Use of simple mathematical tools and inspection of more complicated ones contributes to the same end. Slow pupils can experience a widening of conceptions by taking part in trips to inspect the uses of mathematics in buildings and in industrial plants. Their reading vocabularies can be sufficiently developed for them to appreciate and understand the quantitative expressions commonly found in newspapers and magazines. If the school is equipped with a good mathematical display and the material in it is properly explained, the backward pupil may gain a greater insight into the role of mathematics in civilization than is sometimes suspected.

An outline of work for grades 7, 8, and 9 that carries out the ideas set forth above is given in a chart as Appendix VI. A course for slow pupils that is on the whole in agreement with what has been said here has been reported by Eisner as having been made by The Association of Teachers of Mathematics in New York City.[19] It is built around four large topics, with subdivisions as shown below:

I. The Social Uses of Arithmetic
   Recreational and leisure activities.
   Health activities.
   Thrift.
   Investments.
   Keeping and interpreting accounts.
   Paying taxes.
   Providing for future needs and emergencies.
   Solving miscellaneous family economic problems.

[19] Eisner, Harry. "The Challenge of the Slow Pupil." *The Mathematics Teacher*, Vol XXXII, pp. 9-15, January, 1939.

II. Interpretation and Visualization of Quantitative Data
- Learning of concepts of commonly used quantitative units.
- The comprehension of very large and very small numbers.
- The use and interpretation of tables
- The construction and interpretation of graphs such as found in newspapers and magazines and relating to safety, budgeting, social trends.
- The drawing of simple inferences from statistical facts

III. The Uses of Geometry
- The appreciation of geometric forms around us.
- Measurements of lengths, angles, areas, volumes with instruments.
- The concept of approximate nature of measurement.
- The solution of direct mensurational problems as applied to familiar objects in life.
- Indirect measurements including scale drawings and simple numerical trigonometry.
- Construction of simple geometric figures with instruments.
- The experimental discovery of common geometric relationships.
- The locus.

IV. Algebra as a Tool of Thought
- The use of the formula in geometry and arithmetic.
- Signed numbers.
- The extension of arithmetical operations to include literal numbers.
- The solution of simple verbal problems with linear and pure quadratic equations.

The outline of Appendix VI or that just given could safely be followed by a school system which does not wish to build its own program. Experimental programs of various school systems have been described more or less fully by different writers.[20] In some cases they are connected with vocational work, at least as a means of motivation.

MATHEMATICS FOR SUPERIOR PUPILS

In dealing with brighter pupils, especially in the lower grades, opinion has been divided concerning the advisability of enrich-

[20] In an article, "An Adjusted Curriculum for the Dull-Normal Pupil," *Occupations*, Vol XVII, pp 34 39, October, 1938, Mary P. Cooie reports on programs for ten large cities in the country

ing a course by additional topics and teaching the essentials while accelerating the pupil's progress through school. Because children appear to develop more naturally and have better mental health if they are in classes with children of approximately their own age, and for other reasons as well, administrators recently have appeared to favor the plan of keeping bright children in their own age groups and providing for them an enriched program of studies that will challenge their capacities.

Although the two programs set forth in earlier chapters had in mind average pupils through grade nine, they were quite complete, and some optional material was indicated. By using special works on enriching mathematical instruction,[21] and by referring to somewhat exhaustive textbooks, the teacher should have no difficulty in providing for the needs of either a uniformly superior class, or the superior members of a class with whom some plan of differentiated assignments is being employed.

The programs given for the upper years in the preceding chapters were themselves quite extensive, and it is not likely that additional material will be needed except for very able pupils. One way to provide for special pupils is to direct them into parts of analytic geometry and the calculus that were not suggested in the outline; a high school library should have textbooks that are necessary for this purpose If such instruction is not desired, superior pupils can be given a broadened knowledge by study of some of the following topics:[22]

In geometry
- The notion of continuity
- The three famous problems of antiquity.

[21] A work especially to be noted in this connection is *Enriched Teaching of Mathematics in the Junior and Senior High Schools*, by Maxie Nave Woodring and Vera Sanford, revised edition, Bureau of Publications, Teachers College, Columbia University, ix + 133 pp., 1938. The various compilations in this book can be of the greatest use to the mathematics teacher

[22] The topics are taken largely from lists in *Educating Superior Students*, pp. 197 198; American Book Company, New York, 1935. A number of the topics given in the work referred to are found in the outlines of previous chapters in this Report.

Parallelism and infinity
Non-Euclidean geometry.
Higher dimensionality
The theory of limits.
Study of the foundations of geometry.

In algebra
Generalization of the function concept.
Maxima and minima problems.
Mathematical induction.
Elementary number theory.
Scales of notation.
Determinants.
Statistical theory.

In trigonometry
Spherical trigonometry.
Applications to astronomy and navigation.

An opportunity room for superior pupils is helpful for individual and special study, and has been provided by some schools When this means of guidance and aid is not available, the problem of dealing with the special study of superior pupils becomes a responsibility of different teachers.

Other types of mathematical and related readings than those mentioned are possible,[23] and able pupils can also be given instructive activities in connection with mathematical clubs and school publications For all such projects to succeed it is necessary to have the school library properly equipped.

It is not desirable to extend study to new topics at the expense of thoroughness. Pupils with able minds often believe the quick perception that comes to them is complete understanding, when it is not. In an earlier chapter of this report it was stated that mathematics is an admirable study to reveal the great difference between superficial understanding and mastery, and mastery was indicated as an objective of education. Able students should be guided into the habit of thoroughness, and dilettantism should not be encouraged Many parts of mathematics

[23] The work of Woodring and Sanford, cited on p 145, will be found quite as useful in the matter of bibliographies for the upper grades as for the lower ones, as its title indicates.

must be repeatedly reflected upon and must be considered from many viewpoints before they are adequately understood. Similarly, many of the processes must be employed again and again before they can be assimilated and used with the readiness that a skillful person manifests. An immature mind may find much glamour in what is new, but the trained mind experiences great satisfaction in penetrating more deeply and searching for unsuspected relations among familiar things. Some students go on to university or college with a wide variety of topics in mathematics studied in high school but with limited insight and inadequate technique. A searching question bewilders, and an involved problem is beyond them. Although it is imperative that high school programs should provide plenty of material for able classes or for able students in ordinary classes, the importance of thoroughness should not be forgotten. However much it may be necessary to relax from this ideal in the case of pupils with moderate abilities, it should be kept constantly before those who are superior.

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CHAPTER VIII

MATHEMATICS IN THE JUNIOR COLLEGE

"I put them aside to finish later in the year, and in the meanwhile, deserving, as I thought a little real restful luxury, devoted myself to Differential and Integral Calculus."
—DE MORGAN, IN JOSEPH VANCE.

DEVELOPMENT OF THE JUNIOR COLLEGE

THE rapid spread of the junior college reminds one of the development a century ago of the high school after its first appearance in 1821 as the English Classical School of Boston. Although the junior college is especially prevalent in the West, its origin is to be found in the East and the Middle West. The Bradford (Massachusetts) Junior College and the Joliet (Illinois) Junior College both date from 1902, and of the junior colleges now operating, they are apparently the first institutions to bear the title.[1] Today there are more than 550 junior colleges in forty four states and the District of Columbia. California has 57; Texas, 38; Iowa, 37; and Oklahoma, 32. As to origin, two classifications are to be noted. In the first place, some four-year colleges which were not especially flourishing have contracted their programs and concentrated on the first years of college work.[2] In the second place, junior colleges as such have been founded, both as private institutions and as public schools. Of the private schools some are denominational and some are not; of the public schools

[1] The school in Bradford had existed for a century and that in Joliet for a year before the present names were adopted. See "Junior Colleges," *Bulletin*, 1936, No. 3, U. S. Office of Education, Washington

[2] In this connection it may be noted that President Harper of the University of Chicago, who is often referred to as the "father" of the junior college, gave no less than six reasons why a small ineffective college should drop senior work and become a junior college Cf "Junior Colleges," *Bulletin*, 1936, No. 3, U S. Office of Education, Washington, p 19.

some are supported by a city, some by a district, and some by a state. No small number of junior colleges have closed their doors, and on the other hand some have merged with standard universities or colleges, or have been reorganized as four-year institutions.[3] Although junior colleges are organized differently, some giving only one year of work, and some having preparatory work connected with them, a decided majority of them are two-year institutions.[4] With increasing general public support the junior college is rapidly becoming an important part of the system of public education; and, if present trends continue, it will constitute the thirteenth and fourteenth grades of the public school. Thus, it appears appropriate for this Report to deal with the junior college as falling within the field of secondary education.[5]

The recent tendency of the secondary school system, as here conceived, to expand upwards has created a somewhat new viewpoint toward the entire question of the readjustment of the administrative organization for earlier years. Although the two most common types of organization for the elementary and complete secondary school system are at present the 8–4–2 plan and the 6–3–3–2 plan, it may be that neither of these plans is the one that will most generally prevail in the future, at least in many cities. There are strong advocates of a 6–4–4 plan, an organization in which there are six years of elementary school, four years

[3] *Ibid*, pp 17, 18, 19, 22. Some of the institutions that have closed were founded before the Bradford and the Joliet schools, and the bulletin referred to makes the following statement, which shows the impossibility of any perfectly accurate account. "Many have blossomed and died before their names could be inscribed in any college directory"

[4] According to the "Junior College Directory" of the *American Association of Junior Colleges*, Washington, 1939, there are at present 408 two-year schools

[5] The question whether the junior college should or should not be regarded as part of the secondary school system has been much debated Arguments pro and con can be found in Eells, Walter C. *The Junior College*, Chapter XXIV Houghton Mifflin Co., 1931. The statement in the text is not intended to align the Commission strongly on the affirmative side of the question. Eells, who prefers to consider the junior college as constituting *collegiate* rather than *secondary* education, says he is not disposed to press the matter of terminology

of high school, and four years of junior college or college. It is not within the province of this Report to go into the advantages and disadvantages of such a plan.[6] The discussion deals only with the two-year type of junior college, which, separated completely from a high school, offers work only for grades 13 and 14. As noted before this is the most prevalent type of junior college On the other hand those junior colleges that constitute the lower divisions of universities are also not under consideration. How the programs of such institutions should resemble, and how they should differ from, the programs of strong independent junior colleges is an interesting and important problem, but one which cannot properly be gone into here.

FUNCTION OF THE JUNIOR COLLEGE

The junior college plays the double role of a terminal school and a preparatory school. As the most advanced part of the secondary system it offers an educational terminus for many students that is comparable to college graduation of a century ago. As a preparatory school it provides for other students the preliminary training required for entrance into the upper years of

[6] The question of the four-year junior college is discussed at length by Eells, *op. cit*, pp 672-719 He himself does not favor the arrangement and, for the sake of fairness, sets forth supporting arguments by quotations from its advocates. Eells analyzes the disadvantages of the plan, and after refuting arguments in its favor, devotes a chapter (pp. 720-749) to a discussion of the two-year junior college, which is the one he himself endorses.

In 1925 Harry L Boardman, in a Master's thesis at the University of California entitled *Separation of Junior College from High School*, presented some interesting aspects of the question He showed that heads of junior colleges in California were in general strongly in favor of segregation of grades 13 and 14 from the lower grades Boardman also presented the results of a study designed to show whether graduates of junior colleges that are segregated from a high school had better success in subsequent university work than graduates of junior colleges that are only partly segregated, or not segregated at all He found practically no difference in the achievements of graduates of the three types of schools.

Junior colleges differ as to the type of certificate they give for satisfactory completion of their programs The A A, Associate *in* Arts, or Associate *of* Arts, is granted as a "title" by many junior colleges, and as a "degree" by a few. See "Junior Colleges," p 51, where there is given a quotation from a study made by Doak S Campbell in 1934, which shows that at that time 49 different titles were being given by junior colleges

a four-year college or a university. In the beginning of the junior college movement most of the students were of the preparatory type, but in the last ten years there has been a significant increase in the number of students intending to end their formal schooling with graduation from a junior college. For example, in the Pasadena (California) Junior College in 1926 there were two of the terminal and 43 of the preparatory type of student out of an enrollment of 45; in 1935 there were 318 of the terminal and 192 of the preparatory type out of an enrollment of 510[7] It would seem that the 6–4–4 plan encourages more students to remain after the twelfth year, since entrance into a new school is not involved. A wider adoption of such an organization would therefore increase the tendency, already well under way, toward a preponderance of the terminal type of student in grades 13 and 14.

The junior college should offer a range of instruction that will meet the needs of both preparatory and terminal students, and should provide for (*a*) the first two years of pre-professional education, given in a manner acceptable to the four-year colleges and universities; (*b*) two years of general and liberal arts education, suitable both for terminal students and for those who continue their study; and (*c*) semi-professional education for which there is a community need. A semi-profession is here defined as a vocation for which two years of post-high school education are needed and are adequate

The courses that the junior college offers in any field should be based upon the purposes of its students and upon community needs. They may be limited considerably by the character of the high school preparation presented by entrants and by inadequacy of community resources, and they will doubtless be affected somewhat by the type of organization of the school itself. A mere growth of enrollment alone is likely to result in an in-

[7] *Junior College Journal*, Vol V, p 37, 1934 35 It is to be noted that the Pasadena school, being of the four year type, is not representative, so that figures for it may not be typical Enrollments in grades 13 and 14 alone are included in the figures above

creased number of vocational preparatory curricula, some of which may in turn call for the development of somewhat specialized courses in mathematics and other basic subjects.

PRESENT JUNIOR COLLEGE MATHEMATICS

Mathematics courses in the junior college have generally been patterned after the lower-division courses in the universities.[8] An examination of 352 junior college bulletins and mathematics textbooks in use in junior colleges in 1939 revealed the typical offering to be: intermediate algebra (offered by 146 junior colleges), college algebra (248); trigonometry (295); and calculus (176). There were 143 combined courses for freshmen. Elementary algebra, plane geometry, solid geometry, solid analytic geometry, differential equations, functions of a complex variable, and combinations of these with other subjects were offered, each by a few junior colleges. Mathematics of finance was offered by 69 institutions. Few semi-professional courses were found.[9]

In many junior colleges the courses offered cover approximately 3 to 5 semester hours of college algebra, 2 to 4 hours of trigonometry, 4 to 6 hours of analytic geometry, and 6 to 10 hours of differential and integral calculus. In some junior colleges the courses are not organized under these names but are designated as Mathematics I, Mathematics II, Mathematics III, and the like. Such courses attempt to avoid arbitrary divisions that have seemed to many college teachers both artificial and undesirable. However, a careful examination of these "unified" courses is likely to show that the student who takes from 15 to 25 semester hours of mathematics in "unified" courses will have studied much the same set of topics and to approximately the

[8] Hills, Justin F. "Junior College Mathematics," *School Science and Mathematics*, Vol XX, pp 880-885, 1929

[9] Hannelly, Robert J. *The Mathematics Program in the Junior College.* Unpublished Doctor's dissertation, University of Colorado, 1939. See also Calvin, Funk, *et al. Report of the Mathematics Committee of the California Junior College Association.* Mimeographed materials. California State Department of Education, Berkeley, 1935.

same extent as he would have if he had taken that number of semester hours in the usual courses in algebra, trigonometry, analytic geometry, and calculus No implication is here intended as to which arrangement of material is to be preferred, nor is it implied that a completely satisfactory unification could not produce materially different results

DIFFERENT MATHEMATICAL PROGRAMS NEEDED

Since the mathematical courses now offered by the junior college are in the main patterned after university courses prescribed for pre-professional training, they presumably meet the needs of specialist students Any question as to modifications that might be advantageous for the pre-professional group falls within the field of university rather than secondary education, and is thus outside the scope of this Report. Specific mathematical requirements for advanced college work, particularly in scientific and engineering courses, must be fulfilled if the progress of the student is not to be interrupted. To the extent to which a junior college serves a pre-professional clientele, it may properly include the existing type of course among its offerings.

As already noted, however, a large majority of junior college students are of the terminal type. Moreover, a large proportion of the minority who constitute the preparatory group have a general or liberal arts interest. Thus only a very small part of all junior college students are of a pre-professional type.

In setting up a mathematical curriculum suitable for specialists, it seems to have been assumed tacitly that students of the terminal type, as well as those intending to proceed with more advanced work in such subjects as languages and literature, journalism, and social studies, had already received sufficient mathematical instruction in the high school. Or, it may have been taken for granted that all students who studied mathematics should be given the same material, although naturally some of them would proceed further than others in the established sequence of mathematical courses.

The validity of these assumptions is open to serious question. Any person intelligent enough to graduate from a junior college presumably should be able to carry successfully a course that is planned to give both an insight into the nature of mathematics and an appreciation of its wide and growing importance in modern life. If he does not take such a course, there is a strong likelihood that he will bar himself permanently from understanding some of the most significant and distinctive phases of our present civilization and culture.

Moreover, different groups of college students, with different purposes, have different types of mathematical needs. Non-specialists, particularly those having a general or liberal arts interest, need very little command of the higher techniques; but they do need the sort of insight and appreciation just mentioned—in short, familiarity with the cultural significance of mathematics. Students looking toward a semi-profession or other vocation may also need special mathematical courses different from those designed for pre-professional training. In discussing different types of needs it will be convenient to classify all students outside the pre-professional group as either Semi-professional, including vocational, or Academic

COURSES FOR SEMI-PROFESSIONAL GROUPS

Semi-professional curricula are feasible only in large municipal junior colleges or in those small junior colleges of the district-and-union-of-districts types that are situated in areas in which one or more common semi-professions are practiced, e.g., in rich agricultural or mining regions. Such curricula, not all of which involve courses in mathematics, are in operation in a number of special fields, mostly connected with business or engineering. For semi-professions relating to business, appropriate mathematical work includes computational methods, commerce-algebra, mathematics of finance, and statistics. More detailed suggestions will now be given relating to commercial and vocational groups, followed by a remark on cultural interests.

Commercial Group. The students of the junior college in this group should have an opportunity to continue their work in commercial mathematics.

The work should be somewhat broader than high school commercial or business arithmetic, though it must fit the student's preparation. Some students probably will be deficient in algebra, and since this subject is fundamental to financial mathematics, attention must be given to it early in the course After a foundation in algebra has been laid, questions involving simple and compound interest, true discount and present value, annuities, sinking funds, and amortization, and related problems can be studied intelligently. Detailed recommendations with regard to the course do not seem necessary, standard textbooks are available and teachers should be able to develop materials suitable for special situations. The use of logarithms and the slide rule, as well as computing machines, is recommended.

Vocational Group. Students in this group, many of whom may be graduates of vocational high schools, are likely to have their objectives rather fully developed. They will represent many fields of specialization, and in the same class there may be found future carpenters, electricians, plumbers, printers, masons, and mechanics. In fact, unless the junior college in question is so large as to allow separation into more homogeneous groups, such students must receive common instruction.

There can be no question that the vocational group can profit from the study of mathematics and its applications in their proposed vocations Decision as to the mathematics best suited for the purpose must be reached as the result of a careful analysis of the vocations in question. With the growth of the junior college movement the field will become one of the most fertile for the experimentally-minded teacher Elementary mathematics of engineering, including the strength of materials, might constitute one approach. The study of stress and strain, force, friction, beamed structures, and so on has direct applications in all of the

vocations, with the exception of printing. The printers constitute a somewhat special group. They are more concerned with questions of symmetry, layout, and so on, than with problems that occur in construction work.

Although the vocational group or others in the semi-professional group may express a primary interest in things that are practical, we can hardly assume that only a few members of either group may be appealed to by other phases of mathematics. Matters that are strictly "practical" are likely to interest a person only if they lie in his own field of activity, while ideas and results not so narrowly restricted may stimulate the intellectual curiosity or satisfy the aesthetic sense of an alert mind, irrespective of vocation. So long as we see men in vocations desiring to have homes that have taste and charm as well as homes that are comfortable, we should not doubt that a group of vocational students can be appealed to by the aesthetic nature of mathematics. The subject can be distinctly cultural for them, and it is one of the primary responsibilities of the mathematics teacher of such a group to correct any false impression the students may have that the subject exists simply because it is useful in a narrow bread-and-butter sense. In short, many vocational and other semi-professional students may develop an interest in the sort of course now to be suggested as appropriate for liberal arts students.

MATHEMATICS FOR THE ACADEMIC GROUP

The academic group of students constitutes one of the most perplexing problems of the junior college, in so far as mathematics is concerned. It will be heterogeneous with respect to background; most of the members of the group will have had no more than the equivalent of a year of algebra and a year of plane geometry, while there will be some who have had mathematics through grade twelve. The students whose mathematical study has lapsed should be given an opportunity to take courses different from those of the last two years of high school, which

they already have declined. On the other hand students who had mathematics throughout the high school obviously should be offered something with a new flavor By no means should the work offered the academic group of terminal students be merely customary high school mathematics slightly glorified. Great care should be used in so planning a course that it will give the appreciations and understandings of the nature of mathematics and its accomplishments that may form an important part of the cultural outlook of the well-educated layman.

An important consideration arises at this point Few students who take mathematics in the four-year college go beyond the calculus of the second year, and the majority who take mathematics stop with one year of the subject. The upper year courses are composed largely of mathematics majors, and of other students who require advanced work It follows that junior college courses suitable for the academic group of terminal students should also be suitable for the majority of preparatory students Up to the present time, however, colleges and universities have given a very restricted type of mathematics offering for the first two years, though there is a tendency now to design survey courses for the general student. This trend makes it appear likely that the problem of transferring credits from the junior to the senior college in the field of mathematics may be liberalized

The Commission believes that four-year colleges should give recognition to strong survey courses in junior college, whether or not they themselves offer such work in their first two years. If this practice is followed, many preparatory students of a junior college may be well served by mathematics courses set up especially for the academic terminal students of the school.

The Commission does not wish to endorse only one type of course. Two alternative types will be outlined, with the understanding that others are feasible.

Basic General Course. The first type is designed for a five hour course throughout a year for an able group of terminal-type students whose high school mathematical preparation

includes but one year of algebra and one year of geometry, or the equivalent. For a class of medium ability and preparation some topics would need to be shortened or eliminated; for a group of yet lower ability enough material is here suggested for a two-year course. Flexibility is a matter of primary importance, since one of the problems of the junior college is the adaptation of a course to students with different high school preparation.

General Mathematics, Type I

(*1*) Measurement and computation: comparing distances; accuracy of measurement; significant figures, rounding off numbers; use of exponents; laws of exponents, logarithms, computation by logarithms.

(*2*) Elementary trigonometry: historical development of linear measurements, shadow-reckoning used to determine heights; right triangles, similar right triangles; ratios of sides of triangle as functions of the angles, construction of tables for sine, cosine, and tangent; height and distance problems; laws of sines, cosines, and tangents; solutions of oblique triangles; applications.

(*3*) Graphs and equations: graphs of straight lines and circles; graphs of quadratic functions; solutions of quadratic equations; graphical solution of simultaneous line and circle equations; algebraic solutions.

(*4*) Conic sections: definitions of parabola, ellipse, and hyperbola; equations in standard forms; graphical and algebraic solutions of pairs of conics that are easily solved by quadratics; applications of parabola and ellipse.

(*5*) Statistical representation: illustrations of statistical investigations, measures of central tendency, statistical graphs; scatter diagrams, simple correlation, applications.

(*6*) Normal distribution: simple illustrations of chance distributions; curve of normal distribution, applications.

(*7*) Elementary mathematics of finance: installment buying and selling, present worth of deferred payments; building and loan accounts, annuities.

(*8*) Series: arithmetic and geometric series; sum of first n terms of series; infinite descending geometric series, idea of a limit and the meaning of the sum of an infinite geometric series.

(*9*) Derivatives: slope of a straight line; idea of derivative as slope at a point, idea of derivative as velocity of falling body; derivatives of polynomials, maxima and minima; applications.

(*10*) Integration: the integral as the limit of a sum; application to

areas, integration as inverse of differentiation, uses of integration

A similar selection of material, but with more emphasis upon its significance in the social studies, was recommended several years ago by a committee of the Social Science Research Council The list of general topics there suggested—details of which, with a statement of the reasons for the choice of the material, can be found in the published report[10]—is as follows.

"(*1*) Logarithms, with applications to investment.
(*2*) Graphs, as a tool in the study of tabulated data.
(*3*) Interpolation by various methods.
(*4*) Equations and forms of curves
(*5*) Probability and frequency distributions
(*6*) Elements of differential and integral calculus, including partial differentiation
(*7*) Curve fitting and least squares"

In order to equip students to understand these topics adequately, six to nine semester hours were estimated to be needed.

Higher Orientation Course The other course that is suggested resembles the survey courses that have recently appeared in some universities, and that have caused the Commission to recommend a liberalization in the policy of transferring credits The aim of the course is not so much to prepare students to work with mathematics as to give them a broad familiarity with the nature of various parts of the subject.

GENERAL MATHEMATICS, TYPE II

(*1*) The genesis and development of mathematics; origin in problems of mankind; relation to advances of civilization; important role of intellectual curiosity, vast modern extensions through free creative invention of new fields.

(*2*) Euclidean geometry, somewhat critically viewed; its significance as a logical system, postulates and undefined elements.

(*3*) Non-Euclidean geometries; Bolyai-Lobatchevsky geometry, Riemannian geometry

(*4*) Number successive generalizations, symbolic treatment, numerical computation; simple illustrations of number theory.

(*5*) The group concept; elementary illustrations of finite groups.

[10] *American Mathematical Monthly*, Vol. XXXIX, pp 569 577, 1932

(*6*) Classes: correspondence; types of order, transfinite number, mathematical induction

(*7*) Functions: varieties; use in studying scientific laws; periodical phenomena; some ideas and techniques of coordinate geometry.

(*8*) Limits, occurrence in familiar concepts, derivatives; integrals

(*9*) Statistical concepts, elementary ideas; probability, and the distribution of errors; illustrations from biology, medicine, the social studies, psychology and education, technology and other fields.

(*10*) Further mathematical aspects of the physical sciences: use of differential equations in studying phenomena, relativity, and other modern physical theories.

(*11*) The nature of mathematics, rival views; nature of the foundations; significance as a system of thought, relations to philosophy, aesthetics, and the sciences.

The outlines above are meant to give only a first suggestion of the kind of courses that should be worked out in the interest of the terminal type of junior college student.

In earlier chapters, in connection with the work of grades 9 to 12, the importance of instruction in the history of mathematics was pointed out. The courses outlined above will become more interesting, and may take on new significance, if they are so presented that students gain a clear impression of mathematics as a constantly growing subject. If historical material is properly presented, mathematics gives a student a look at the centuries as well as helps him to understand contemporary life.

CHAPTER IX

EVALUATION OF THE PROGRESS OF PUPILS

". . and whether he stood fortieth or ninetieth must have been an accident or the personal favor of the professor. Here his education failed lamentably. At best he could never have been a mathematician; at worst he would never have cared to be one; but he needed to read mathematics, and he never reached the alphabet"

—HENRY ADAMS, THE EDUCATION OF HENRY ADAMS

A GENERATION ago the testing movement as we know it today was in its infancy. Although teachers thought of testing as a part of their work, their purposes and the methods they used may now be regarded as rather inadequate. Almost all teachers believed that the chief purpose of giving tests was to furnish a basis for assigning grades In framing test questions and directions characteristic words were *who, what, when, where, define, describe,* and *discuss.* The most important qualities which a pupil needed in order to respond successfully were a good memory and conscientiousness in studying the lessons that were assigned The chief concern of teacher and pupil alike was mastery of the subject matter of the course, while the school as a whole made little systematic effort to determine the extent to which different studies and other school activities contributed toward a general and broad development of the pupil as an individual on the one hand and as a valuable citizen on the other.

As time has gone on a transformation has been under way Many teachers have broadened their conception of the purposes of testing, and gradually the emphasis is shifting to a different set of characteristic words. The cue is now more often *why, how, explain, interpret* There is more concern that the pupil

understand as well as remember what he has learned. Probably, however, the most striking development has been the large number of so-called objective tests that have been published and used. Unfortunately it often seems that the authors have been so interested in making use of new testing techniques that they have neglected basic questions relative to the validity of the tests. In spite of this, however, the tests have revealed many significant facts about achievement, and have stimulated the leaders of the movement to re-examine certain assumptions hitherto uncritically accepted. Finally the scope of testing has been greatly extended, and an ever larger group of teachers has become concerned with the evaluation of more than subject matter achievement. They recognize that mastery of various bodies of subject content is but one aspect of education, and they are attempting to evaluate the development of interests, appreciations, and other characteristics of personality to which the schools are increasingly directing their attention. In this connection it is important to note that evaluation means more than the giving of tests or examinations: the term is used to refer to any method of obtaining and interpreting evidence about the development of pupils. In the process, the discovery of relationships among data bearing on different aspects of development is very important.

The purpose of this chapter is to call attention to some of the problems in the field of testing and to encourage investigations that will contribute to their solution. It is obviously impossible to give within the confines of a single chapter more than a brief summary of recent developments. The Commission has therefore chosen to consider only a few basic topics. It will discuss the purposes of evaluation, some limitations of the types of tests now commonly used, some advances that have been made in recent years, and some suggestions for improving this phase of instructional activity. Much of the discussion will be quite general and will apply not only to the teaching of mathematics but to teaching in other fields as well.

THE PURPOSES OF EVALUATION

Primary Purposes. Many teachers would assert that the chief purpose of testing is to provide a basis for assigning marks, perhaps a deeper analysis is not to be expected from those who stagger under a teaching load of five or six large classes a day. But tests are given for many other purposes, among which are the following: to maintain standards, to select and reject pupils, to discover strengths and weaknesses of individual pupils or of the class as a whole, to provide a powerful incentive to study, to furnish a convenient method of instruction, to stimulate or even enforce improvement of teaching, to afford a basis for the appraisal of teachers and departments, to serve as a basis for accrediting schools and colleges, to furnish data for educational guidance, to accumulate materials for research.[1]

Some of the purposes noted are formulated from an administrative point of view It is the administrators, for example, who are most interested in using tests as a means of evaluating the effectiveness of teaching. The use of tests to discover the difficulties of individual pupils seems to spring from a different type of purpose--it focuses more directly upon the learning situation and the pupils Thus it is possible to distinguish at least two types of purposes which differ in point of view. The first is primarily concerned with evaluation of the educational status and progress of individual pupils; the second is primarily concerned with evaluation of the school as an institution. Both types are important, but it is quite obvious that evaluation of the second type depends upon evidence obtained from evaluation of the first type. Under the emerging concept of evaluation, attention is focused primarily upon the pupil, the fundamental purpose being to obtain a comprehensive characterization of him as an individual and to discover the effects of school experiences upon him

[1] Hawkes, H F, Lindquist, E F, and Mann, C R (eds) *The Construction and Use of Achievement Examinations* (Houghton Mifflin Co, 1936) See Chap. IX, "Uses and Abuses of Examinations," by Max McConn

Evaluation, when approached from the point of view just stated, is analogous to the procedure of the biologist examining the internal structure of a minute plant or animal. Thin cross-sectional slices are viewed under the microscope. By studying the variation in structural relations from slice to slice, the biologist is able to reconstruct the organism mentally and to perceive the interrelations of various parts. Similarly, teachers give a variety of tests and also use other means of arriving at judgments about specific aspects of learning and personality growth. The real task of evaluation, and the real purpose of testing, is to piece together the data of varied types and from many sources into a composite picture of the individual.

When evaluation is viewed in this general way some of the weaknesses of many current practices become more apparent. Thus the types of tests ordinarily given are seen to be seriously inadequate in several respects. One recognizes the need for more comprehensive and reliable testing programs in order to avoid giving a distorted picture of the individual A few numerical or letter "grades," representing often little more than ability to recall information about school subjects, are hardly sufficient to serve as a basis for knowing what a person is really like, or what effect the school, or even a particular study like mathematics, has had upon him The necessity of careful interpretation of test results becomes very clear, and one begins to understand why it is important that results on a particular test, or in a particular subject matter field, should not be viewed in isolation but in relation to results from many other tests and evidence about a number of different aspects of school experience.

The acceptance of the fundamental purpose of evaluation stated above carries with it a number of other purposes as corollaries Thus it includes the discovery of the strengths and weaknesses of individual pupils—it implies that the testing program should be *diagnostic* in nature. If an evaluation program is in fact diagnostic, and if it is granted that the school should

attempt to develop the talents and remove the weaknesses of pupils, then at least an elementary form of educational guidance is indicated. This Commission, of course, cannot undertake to describe how a guidance program may function, but it can endorse the proposition that guidance should be based upon data obtained from reliable tests. If, for example, one obtains valid evidence that a pupil has talent and interest in mathematics, the suggestion that he should take further work in the subject may be justified. On the other hand it may be discovered that he does poorly in mathematical work—in fact, he may lack some of the fundamental mathematical understandings and skills that are vitally important for successful home living or competence in many occupations. In such cases it does not follow that he should be advised or permitted to avoid mathematics. It seems rather that steps should be taken to provide instruction of a type designed to remove the weaknesses and fit him for competent citizenship. Such considerations strengthen the belief that the basic purpose of testing is the construction of a comprehensive description of the individual. We shall have more to say about these matters in later sections of the chapter. Since, however, many other purposes of testing have been mentioned by different writers, some discussion of them may be warranted at this point.

Comments on Other Purposes. There is one purpose of testing that looms large in the thinking of many teachers. Tests are said to be necessary for the *maintenance of standards.* According to one writer, this "seems to mean either one or both of two things: the imposition and enforcement of a prescribed curriculum, or the enforcement of some minimum degree of attainment." In practice, it means that schools, state departments of education, or colleges acting through an examining agency such as the College Entrance Examination Board, require pupils to secure a certain minimum score on one or more examinations in order to achieve some desired educational status—promotion to another grade, a diploma, entrance to college, and so

on. There are certain types of situations in which the protection of society makes such examinations a desirable and justifiable hurdle for the candidate to surmount[2] The licensing examinations for entrance to the legal, medical, and teaching professions are cases in point. This also applies to examinations for admission, promotion, and graduation set by professional schools; but there is some question as to how far such a justification may be extended What position is to be taken on the maintenance of standards in the secondary school now ministering to the adolescent population as a whole? In seeking an answer to this question it is helpful to examine some of the assumptions upon which the notion of standards-enforcement is based.

The most obvious assumption is that for any given field there exists both a well-defined body of subject matter and a standard of attainment that pupils may reasonably be expected to meet. It happens, however, that testing programs based on this assumption have themselves provided evidence throwing doubt on its validity. They have, for example, revealed the wide range of differences in achievement that exist between individuals, classes, and school systems.[3] When the same test is given in different schools it is not uncommon to find that the lowest score recorded in one school may exceed the highest score made by any pupil in another. In a recent state testing program it was found that of two schools serving the same type of community and situated only a few miles apart, one stood at the top and the other at the bottom in the distribution of mean scores Another source of confusion is the lack of comparability between tests given in successive years; rarely is there any evidence that a given score on one test represents the same achievement as the same score on another test in the subject given the following year. Under these circumstances the insertion of a few items that differ in difficulty from those on previous tests may bring

[2] Hawkes, Lindquist, and Mann, *op cit*, pp. 452 ff.

[3] Eckert, Ruth E "Realism in Higher Education" *Educational Record,* Vol. XIX, pp 86-104, 1938.

about a change in the standard. It has also been found upon analysis that, on the average, pupils do well on material that has appeared in a number of previous examinations, although it was not included in the course of study, and that they do poorly with material listed in the course of study but not included in previous tests. Such findings seem to make untenable the assumption that uniform standards of subject matter content and attainment exist in practice.

A second major assumption often made in connection with testing that purports to foster maintenance of standards is that the tests used yield reliable measures of achievement. This also is often highly questionable. Enforcement of standards in the sense here used depends upon the establishment of some critical or minimum score. Passing over the point that this is usually done quite arbitrarily, attention is to be called to the fact that while the tests used may satisfactorily determine both high and low achievement, they may lack sufficient precision and reliability to place border line cases accurately. Yet in order to maintain standards in the strict sense this accuracy is essential.

At present it is not uncommon for a teacher to issue the arbitrary dictum "You must get a mark of so-and-so per cent on this test, or else . . . ." The alternative may be to receive a "failure" in the course, to repeat the course, to drop out of school, or any of a number of other possibilities. This sort of treatment may be regarded as a first rough approximation to a method of maintaining standards. When one of the alternatives becomes an actuality it so frequently results in psychological maladjustment that some educators are supporting a policy under which pupils are advanced into the next grade or course irrespective of past achievement. There are many observers of the educational scene who assert that such a policy means a complete breakdown of the concept of standards and leads to educational chaos.

Recognition of the dubious validity of assumptions like those mentioned above and of the difficulties into which they lead is

resulting in modifications of the ways in which test results are used in maintaining standards.[4] There is a growing tendency to believe that escape from the difficulties lies in the direction of devising not only better methods of measuring but also better methods of analyzing, recording, and reporting achievement. Arbitrary decisions, such as those typified by the phrases, "All who get less than 60% will fail" or "Henceforth there will be no failures in this school," must be supplanted by carefully formulated judgments concerning what is best for each pupil. It is particularly important that the welfare of able students be not overlooked because of preoccupation with those who are having difficulty. Tests should be regarded as more than a method of "separating the sheep from the goats." They are to be used as a means of discovering what each pupil needs most in order to gain the full benefit of school experiences. When so regarded, the notion of how tests may be used to maintain standards is extended. Achievement of the highest possible quality is to be sought for all. Tests carefully constructed and used primarily for diagnosis are a means to this end.

It was noted above that from the administrative point of view evaluation is important in order to judge the effectiveness of the school as an institution. The emphasis in this case is not upon the development of the individual pupil, but rather upon whether the teachers are producing the kind and amount of growth that might reasonably be expected of the pupils as a group. This is a justifiable purpose, but in following it great care must be taken to be sure that the judgments are valid All that has been said about evaluation of the growth of pupils applies with additional complications to the evaluation of the work of teachers. Here also, a comprehensive picture is needed. To conclude that a given teacher or school system is ineffective on the basis of the results on a single subject matter test may be as erroneous as the idea one would get of a microscopic organism

[4] See Eells, W C "Basis for a New Method of Accrediting Secondary Schools" *Educational Record*, Vol XIX, pp. 114-142, supplement 11, 1938

from the examination of a single cross section. The teacher may in fact be doing superior work in developing many objectives that the test does not measure.

In this connection it may be well to call attention to certain points often overlooked in testing programs. When the same test is given to a number of different classes (as in state-wide testing programs), it is customary to compute an average for the group as a whole and to compare the average achievement of a given class with the average of the group. When this is done, teachers or classes that fall markedly below the group average are often subjected to censure. As a result, steps are taken to insure that in the future higher class-averages will result In general this is desirable, since it tends to raise the level of the achievement of the group as a whole But it is sometimes overlooked that the average is a statistical measure, and that the abilities commonly measured are, roughly speaking, normally distributed It thus appears that if the pupils in a given class are below normal in other respects—of low intelligence, for example, or the product of poor teaching in earlier work—then one cannot expect the class average to exceed that of the total group. No amount of pressure upon teachers and pupils to raise the averages will bring about a situation in which the average score of every class exceeds the average of the group of pupils as a whole. Here again the only safe basis for judgment of success or failure of teachers is data on many factors that must be studied to discover possible relations existing among them. The criterion of success should be phrased in terms of the growth actually produced in relation to the characteristics of the particular group, rather than in terms of scores made on a particular test and their relation to averages based on scores from large heterogeneous groups.

Summary Considerations of the kind mentioned above and others that cannot be discussed here are bringing about new conceptions of the purpose of testing The use of tests to maintain arbitrary standards and to appraise the effectiveness of

teachers is entangled in many difficulties and subject to abuse. Teachers are becoming more conscious of the fact that the ". . . major utility of examinations is educational guidance. That the developing doctrine of guidance demands, first, diversification of standards and courses and schools, and, second, the general introduction of the methods now available for the study of individuals . . . in examining for guidance we need all known kinds of tests, many of them, preferably comparable tests, and preferably tests used explicitly for this purpose."[5]

The Commission wishes to emphasize in addition that examinations should aim to reveal the effects of all the educational forces acting upon the mental, physical, and personality characteristics of individual pupils.

SOME LIMITATIONS OF CURRENT TESTING PRACTICES

Scope of the Objectives Commonly Tested. How well do the tests now commonly used serve to reveal the effects of school life upon the development of individual pupils? The discussion thus far has implied that no single test is adequate for this purpose But what are the criteria for adequacy? To answer this question in detail would lead to technical discussion beyond the scope of this Report. But we may observe that the answer depends fundamentally upon the statements of the objectives that the schools hope the pupils will achieve. Earlier chapters of this Report have discussed certain educational objectives at some length The real question, then, is this: What sort of evidence do we now obtain about the achievement of these or similar objectives? The first criterion for the adequacy of a testing program must be expressed in terms of the extent to which the tests given yield evidence concerning the achievement of *all* the objectives that are considered to be important.

We must recognize at the outset that until recently few teachers have been concerned with more than subject matter achievement The tests given by mathematics teachers have not

[5] Hawkes, Lindquist, and Mann, *op cit*, p. 478

been designed to measure more than achievement with respect to the subject matter of specific courses. The same remark applies to tests prepared for state-wide testing programs and for entrance to college. This situation exists in spite of the fact that the objectives stated for these courses often emphasize *general abilities*—for example, the ability to interpret data, the ability to generalize, and the ability to reason logically. These objectives, and others like them, are also considered important by teachers of other subjects. The study of mathematics can contribute to the growth of these abilities, but it is not unique in this respect. In stating these and similar objectives the assumption is commonly made that they are general abilities to be applied on any appropriate occasion But types of tests in common use do not take cognizance of this assumption, and restrict the field of measurement by sampling only the application of the abilities in situations ordinarily or exclusively mathematical. In order to satisfy the criterion given above we must either plan to extend the scope of our sampling of pupil reactions to include many other types of situations, or we must suitably restrict the statement of the objectives so that they apply only to situations that are definitely mathematical. The latter alternative is one that few teachers are ready to accept permanently, but we must admit that at present our tests give us little direct evidence concerning achievement of many of the important general objectives of instruction in mathematics.

A careful analysis of both the tests given by classroom teachers and those prepared by extramural examining boards reveals that they are restricted in another way—they measure achievement of only a few of the objectives that are specifically mathematical. The ordinary test places too much emphasis upon the measurement of technical facility and ability to recall specific facts and principles In the field of algebra, for example, a considerable number of test items is devoted to measurement of the ability to perform operations with abstract symbols Skill in adding or multiplying monomials, solving equations, and evaluating

formulas is an important objective of instruction in mathematics, but not the only one. The well-trained mathematics teacher should also be interested in discovering the extent to which the pupil understands the significance of what he is doing. The measurement of this understanding calls for questions that require interpretative answers—statements that explain why one proceeds as he does, that describe the values achieved by performing the process, or that show in what sense the formula or equation involved is a powerful tool of thought. Questions of this type are sadly lacking in most tests, and their absence is the basis for asserting that present-day tests measure only restricted types of objectives even within the field of mathematics.

One reason for the existence of this situation grows out of certain theories of test construction. Early work in the field revealed that the scoring of test papers was often a highly subjective process—the same paper when marked by different teachers was assigned a wide range of scores. Following this discovery, test experts began to seek techniques of scoring that would result in a given paper receiving the same score when marked by several competent teachers. This led to the modern objective type of test and the widespread use of short-answer techniques, such as the "true-false" and "completion" type of test item The desire for *objectivity* stimulated a concern for techniques that tended to obscure more basic questions of validity. The methods used proved effective in the measurement of skills, the ability to recall facts, and similar objectives, but they appeared to be unsuited for the measurement of understandings, appreciations, and other objectives often called "intangible" The assumption that objectivity requires the use of the better known techniques has prevented some examiners from concentrating on the ability to be measured and devising a technique to suit. This has tended to stifle measurement of achievement of many important objectives The assumption referred to above is not, however, an essential one, and recognition of this fact opens the door to

investigations in which the determination of suitable short answer techniques becomes subsidiary to other more basic questions.

There are, of course, other reasons for the failure to attempt to measure certain complex abilities, understandings, and appreciations. Perhaps one of these reasons is the assumption that if a pupil shows satisfactory achievement with respect to certain rather specific abilities he has achieved the more general objectives. No one doubts that in order to think clearly and succeed in mathematical work a student requires certain essential facts and skills. If these abilities are highly correlated with achievement of the more general objectives, it would be unnecessary to test directly for the latter. But to settle this question it is necessary to devise valid means of measuring the more complex abilities, understandings, and appreciations, and this has rarely been done. However, a similar assumption with respect to the objectives of a number of different college courses has been tested and it has been found that the assumption is not entirely valid.[6] Similar findings in the field of secondary mathematics tend to confirm the observations of teachers that pupils who apparently know the facts and can perform the operations mechanically may lack real understanding and comprehension of the significance of the processes.[7]

Testing in mathematics must find ways of measuring achievement of many objectives in addition to those dealing primarily with recall of information and operational skills. Many excellent tests exist for the measurement of these abilities, and for certain other objectives of restricted types Study of test exercises indicates that in some cases successful responses suggest the inference that the pupils understand what they are doing and why they are doing it But ordinarily direct evidence of this

[6] See Judd, C H, *et al* *Education as Cultivation of the Higher Mental Processes*, Chapter II ("The Relation between Recall and Higher Mental Processes," by R W Tyler) The Macmillan Co, New York, 1936

[7] *Ibid*, Chapters IV and V ("The Number System and Symbolic Thinking," "Algebra, a System of Abstract Processes," by E. R Breslich)

understanding is not obtained. Here, then, is a promising field for investigations relating to instruction in mathematics.

Consumers vs. Producers of Mathematics. There is one aspect of the usual type of mathematics test that is so important it deserves special mention. An analysis discloses that tests seem to be designed primarily to measure abilities required by those who use mathematics from a somewhat advanced standpoint. The number of such persons, however, is relatively small. A far greater number of citizens are, on occasion, consumers of mathematics: they read newspapers and magazines, and they study semi-technical or technical books in which mathematics is used to obtain or explain the findings of investigators in many fields. In such cases, the abilities required are primarily interpretative rather than manipulative.

Mathematical instruction has hitherto proceeded on the assumption that operational abilities are essential to interpretative understanding, or that the latter is a concomitant of the former. This assumption should be tested, and for this purpose tests are needed which minimize technique and concentrate upon interpretation. If it should be found that interpretative abilities can be developed by attacking them directly rather than through the intermediary steps of calculation, then some reorganization of mathematics courses would appear to be in order. The interpretative aspects of mathematics range from very simple arithmetical notions to relatively complex concepts associated with calculus and statistics. Comprehension of the magnitude of large numbers, such as the size of the national debt, is one simple illustration;[8] the interpretation of tabular and graphical data is another. Matters of this kind receive insufficient attention in most mathematics tests

Evaluation of General Objectives Mention has been made above of certain general objectives at present imperfectly tested The illustrations given were all related to the development of

[8] See Judd, C H "Informational vs. Computational Mathematics," *The Mathematics Teacher*, Vol XXII, pp 187-197, 1929

various aspects of clear thinking—for example, the ability to interpret data and the ability to reason logically. But there are many other general objectives of education. They may be classified under various heads, the following being typical: Interests, Appreciations, Attitudes, Effective Work Habits and Study Skills, and Emotional Maturity. The well-trained mathematics teacher can contribute to the development of many specific aspects under these general headings, but growth with respect to desirable goals of these types is rarely measured. It is true, of course, that teachers in other fields have also tended to neglect the measurement of these so-called "intangible" or "ultimate" objectives, in spite of the fact that in certain fields—notably the arts (including music) and English—objectives relating to interests and appreciations are of paramount importance. Similarly, many teachers of the social studies are deeply concerned about the social attitudes of their pupils. During recent years much work has been done toward the development of techniques for evaluation with respect to objectives of the types mentioned As the effectiveness of these techniques increases and the value of the resulting interpretations becomes better known, we may expect similar techniques to be applied by mathematics teachers in order to measure the effectiveness of their instruction with respect to such objectives and to provide a basis for sound educational guidance The construction of tests that measure growth toward "intangible" objectives and that have special reference to mathematical instruction waits primarily upon the recognition of need for them and of the values that would accrue from their use.

SOME ADVANCES OF RECENT YEARS

The Commission has discussed the limitations of present tests at some length in order to suggest directions in which improvement may be sought It might appear from the discussion that little improvement has been made in testing during the past few years. This impression would be erroneous, for the last decade

has witnessed a number of significant developments. A brief discussion of several of these may not be out of place here.

The Report of the National Committee, to which reference was made in earlier chapters, included as an appendix an article by Professor C. B. Upton on "The Testing Movement in Mathematics." That article summarized the status of the testing movement in 1923. In this chapter comments will be made only upon some of the developments since that time.[9]

Among the promising developments since 1923, the following may be noted: the extent to which knowledge of testing methods has increased among teachers in general; the publication of improved tests of various types; the growth in use not only of improved achievement tests but more particularly of tests for other purposes, such as diagnostic and instructional tests The latter development has been facilitated by the appearance of various workbooks that are designed primarily with particular purposes in mind. This movement, already under way in 1923, was discussed and its growth predicted by Professor Upton in the Report of the National Committee on Mathematics Requirements.

In spite of growing recognition that modern methods of teach-

[9] It is not possible to discuss particular tests and techniques in any detail, but the following references are given.

Hildreth, Gertrude *A Bibliography of Mental Tests and Rating Scales.* The Psychological Corporation, New York, 1933.

Buros, Oscar K. *Educational, Psychological, and Personality Tests of 1933 and 1934,* Rutgers University Press, 1935 (Also extended to period 1933-1936, 1936.)

Hawkes, Lindquist, and Mann, *op cit*

Stalnaker, John M "Report on the Mathematics Attainment Test of June, 1936" *Research Bulletin* No 7 of the College Entrance Examination Board

Breslich, E. R *The Technique of Teaching Secondary School Mathematics,* Chapter VII. University of Chicago Press, 1934.

Smith, D. E and Reeve, W. D *Teaching of Junior High School Mathematics.* Ginn and Co, Boston, 1927

Every Pupil Tests, Mathematics, Supplementary Bulletin. University of Iowa, New Series, No 716.

Announcements of Tests The Cooperative Test Service, 500 West 116th St, New York

Ruch, G M *The Objective or New Type Examination* Scott, Foresman and Co, Chicago, 1929

Buros, Oscar K (editor) *Mental Measurement Yearbook,* 1938 Rutgers University Press (Includes reviews of tests and of books on measurement)

ing involve more comprehensive evaluation programs, the greatest emphasis has been and probably will continue to be upon the measurement of achievement. We shall now discuss some of the developments in this particular type of testing.

We have already mentioned the growth in popularity of the workbook. This device has made testing materials available to teachers in a cheap and easily administered form. For the most part the responses of the pupils may be quickly and objectively scored. This development has probably been an influential factor in making teachers acquainted with modern "short-answer" techniques. On the other hand, workbooks have in some cases contributed to formalism. Teachers have sometimes allowed pupils to "fill in the blanks" as a means of disposing of the class period with a minimum of exertion, both mental and physical. The blame for such abuses should, however, rest not upon the instrument, but upon the persons who misuse it. Many of the criticisms of tests in general are applicable to workbooks, but the weaknesses are not necessarily inherent in the devices, and may be eliminated by experimentation and study.

A second significant development of recent origin is the Co-operative Test Service. This service publishes annually a large number of different tests. In the field of mathematics alone the following are available: General Mathematics, Elementary Algebra through Quadratics, Intermediate Algebra, Plane Geometry, Solid Geometry, and Trigonometry The annual forms are approximately equivalent in difficulty, and norms are so computed that comparisons are possible. These tests are wholly objective, using several of the well-known short-answer techniques, and are carefully constructed according to widely accepted technical principles. They are representative of the best modern achievement tests generally available.

Another important development of recent years has been the spread of state-wide testing programs They are usually sponsored by the state universities, or state departments of education, and are provided for the schools of the individual state and

others interested, on a non-commercial basis. The decision to use them or not usually rests upon the individual school or teacher. Carefully constructed (and often experimentally validated before publication) by outstanding teachers or committees in cooperation with experts at the universities, they are a distinct improvement over the average test prepared by classroom teachers. Norms are computed that afford teachers of the state a basis for comparison of their work with that of others in so far as the tests, pupils, and objectives are similar, and to the extent that the tests measure the achievement of these objectives. The programs and tests of the states of Iowa, Wisconsin, and Ohio are representative of this movement.

For a generation the College Entrance Examination Board has been a dominant factor in secondary education. The common criticism that extramural examinations encourage the tendency of teachers and pupils to regard passing the examination as the chief aim of the course, and the tendency of the examination to retard curriculum reform, has often been stated with added force in connection with the College Board examinations. As a result the Board from time to time has changed its requirements. A noteworthy revision in the field of mathematics was adopted in 1935 Beginning in 1936 the Board instituted three new comprehensive examinations designated Alpha, Beta, and Gamma They attempt ". . . to combine the advantages of the longer essay-type, multiple-step question with those of the single-step question. . . "

The following quotation from the report of the Commission which formulated the revision will indicate the spirit of the recommendations.

> In formulating its recommendations, the Commission has been strongly influenced by the wish to leave teachers of mathematics in secondary schools free to guide the developments of their pupils in such ways as seem to them most desirable.
>
> It has been the aim of the Commission not to indicate the content of the mathematics courses which lead up to the examinations in such detail as to hamper the work of the teacher, but to be suffi-

ciently definite in specifying the scope of the examinations to avoid uncertainty on the part of the teacher.[10]

The new examinations are superior in several respects to the former type used by the Board. In the first place, they are much more readily scored in an objective manner; second, they are more general in nature and do not emphasize the traditional divisions into algebra and geometry—in fact, the solutions of some problems require an effective integration of these branches. Finally, there have been indications that the examinations would include items that aim to investigate the candidate's understanding of the nature of logical reasoning. As yet, however, they have not gone very far in this direction

The above discussion of some of the recent developments in the field of educational measurements as applied to mathematics should make clear that continuous improvement is taking place By way of summary we may note that aside from the points mentioned previously the changes have been chiefly in the direction of increased objectivity through the use of short-answer techniques, increased availability of carefully constructed tests, and possibly reduction of the influence that extramural examining bodies have exerted upon the curriculum

In connection with the last point, it may be noted that many educators have expressed the view that the testing programs mentioned above tend to obstruct desirable curriculum changes There is little doubt that this fear is well founded; but it may well be that the tests also prevent some teachers from making unwise revisions and eliminations, and from undertaking experimental programs which are unsound in theory and execution. On the other hand, the evil effects are within the control of those who make the tests and administer the programs and who can use the tests to bring about gradual improvements in methods and materials that would otherwise be deferred for some time. Thus these programs are a potential source of strength, par-

[10] *Report of the Commission on Examinations in Mathematics.* College Entrance Examination Board, New York

ticularly in that they provide data on which to base conclusions and revisions. Real progress is slow when it depends upon random experimenting on the part of individual teachers, but much can be done through concerted action based on reliable evidence.

A PROGRAM FOR THE FUTURE

The preceding sections of this chapter have indicated some of the developments of the last fifteen years, and have pointed out some of the limitations that are recognized to still exist. The obvious program for the next few years involves work toward the removal of the weaknesses of current evaluation practices and the development of new or improved methods. The following discussion will summarize the major points on which attention may well be centered.

Evaluation. The development of means of evaluating achievement of important objectives not now measured is the first essential of the emerging program. This will include a preliminary clarification of the meaning of the objectives, for in many cases they are only vaguely formulated and few descriptions of the specific behaviors involved exist. It will also include the collection of numerous illustrations of situations in which the pupils should exhibit one or more of the specific behaviors in terms of which a particular objective is defined In many cases these illustrations should be actual concrete life situations rather than relatively abstract "problem" exercises The preceding steps are in reality a part of curriculum making Evaluation proper begins when the pupils are faced with the situations and a record is obtained of their behavior—what they do or how they think. Further steps in the process involve the interpretation of the behavior and the attempt to find convenient and reliable short-cut methods of obtaining the record. In some cases familiar types of short-answer techniques may be found satisfactory. But in general we must expect more elaborate testing devices in order to measure the complex behaviors associated with many general objectives.

Methods of Reporting. Revision of methods of reporting achievement is a second essential in the program for the future. In the past we usually have attempted to describe achievement in terms of a single numerical score or letter grade. This assumes that the abilities involved are distributed on a single linear scale If the abilities measured by the tests are quite homogeneous this is a justifiable and convenient practice; but there is little basis for assuming that all of the varied abilities included in the objectives of mathematics are homogeneous. The attempt to lump the different aspects of achievement into a single score often conceals more than it reveals.

In the future we should seek to express achievement verbally in terms of the stated objectives of instruction For this purpose we shall need the results from evaluation devices or tests that measure separately growth toward each important objective. In many cases it will be possible to compute average scores or normal behavior with respect to achievement of the different objectives. We can then describe the status of the individual in terms of a number of different factors. This permits the replacement of a single score by a comprehensive verbal description of an individual. The latter may, of course, include mention of the various scores that are included in the total mass of evidence.

It may be well to clarify these points somewhat. Suppose a pupil has taken tests that measure different objectives. The teacher of geometry might thus have available the scores on tests that reveal:

Ability to recall geometric facts and principles
Understanding of geometric concepts.
Ability to prove "original" geometric exercises
Understanding of the role of definitions and assumptions in an argument.
Ability to recognize and use certain principles of logic in life situations.
Social and economic attitudes or beliefs

Work habits and study skills, including reading comprehension.

Scope and depth of interests.

This list is only a sample of the different types of objectives on which data may be obtained. The last three items are objectives of concern to teachers in other fields, and in some schools the tests for them are given on a school-wide basis. In such cases the teacher of geometry would be responsible primarily for the first five objectives only.

With scores from these tests available, what report might a teacher make about a given pupil? In the first place, it would be possible to report or comment on work habits and study skills, including reading comprehension, and on scope and depth of interests These obviously have a bearing on achievement in most subjects, and particularly in geometry. In the second place, it is possible to comment briefly on achievement with respect to the objectives relating more specifically to mathematical instruction. One pupil may be superior in solving exercises. Another may do well with purely geometric material, but may have difficulty in applying principles of logic in life situations. He may, for example, allow his social and economic attitudes or beliefs to supersede logical principles in arriving at conclusions in these fields. Data relative to the sixth item above would serve to support or perhaps to weaken a teacher's judgment on this point. Still another pupil may understand the role of definitions and assumptions in an argument, but show rather poor understanding of geometric concepts.

Written reports, sometimes in the form of a letter to the parents with a copy retained for the school files, are now being used in a number of schools. The labor of preparing such reports is of course greater than that involved in assigning a numerical or letter grade. For this reason reports are prepared less frequently, say three or four times a year instead of six, but they are more thorough and more revealing. Moreover, the task of accumulating the evidence and preparing the reports may be

distributed over a considerable period of time. When achievement of general objectives is being evaluated, it is not necessary to wait until the end of a marking period to administer tests and prepare reports. When tests measure principally recall of information and skills, a few days or a week or two of instruction may produce appreciable increments to a test score, but improvement is much less likely in a short time in the case of objectives less specific

If data on the different abilities are obtained at intervals, it is possible to describe growth and retention; if in addition data are available on home and community background, previous training, and similar factors, still more significant interpretations may be made. The relation of growth made to initial status may be more significant than the pupil's final status. Thus a pupil who originally ranked very low in several abilities may succeed in reaching the median. If his background and a number of personality traits are taken into consideration, this development may merit high praise, and may represent greater achievement than that made by a pupil originally well above the median who succeeds in maintaining his position but shows relatively little growth. Numerous new methods of recording and reporting pupil progress are being devised and tried. In every case the aim is to furnish to pupils, parents, guidance officers, future employers, and college entrance officials data more revealing than data provided in the usual records and reports

We may summarize this section by saying that a forward looking program of evaluation involves the development of new methods of judging, recording, and reporting the achievement of pupils. These methods should emphasize growth toward the objectives of the educational program and the particular course, and interpretations should take into account evidence concerning many relevant factors that influence achievement.

Administrative Problems. The task of evaluation as here outlined may appear to be so great that the practicality of the

proposals is open to question. This is a pertinent issue, and cannot be taken lightly. The following observations may help to allay anxiety on this point.

In the first place, when general objectives become dominant in an educational program many teachers can contribute to their development and to evaluation of achievement. The teacher of a particular subject, such as mathematics, need not be solely responsible for obtaining all the evidence. A cooperative or school-wide program of testing becomes possible, and the time and work necessary to effect it may be shared by many teachers.

In the second place, the testing does not need to be carried on in a continuous block of time, say at the end of the year, leading to the complete exhaustion of both teachers and pupils. Ideally, it will be distributed throughout the school year, and the keeping of cumulative records will make possible the interpretative comparisons that should be made. The total testing period need not absorb an undue proportion of school time. Some of the time now spent in relatively ineffective but extensive testing of a few objectives may be expended on testing for other objectives, and the results of the tests if properly used should lead to more effective teaching which would compensate for the extra time required. The American schools now spend hundreds of millions of dollars each year on education and the pupils spend hundreds of hours in school. It is not too much to expect that ten or fifteen per cent of this expenditure may well be devoted to the determination of the effectiveness of the remaining portion.

The administrative problems associated with a comprehensive testing program are gradually being solved. One striking example is afforded by the recently invented electrical test scoring machine With the aid of this complex machine hundreds of test papers may be scored in an hour by clerical help, and with an accuracy exceeding that obtained by hand scoring This device may ultimately relieve teachers of most of the drudgery of scoring test papers, and leave them freer to devote their talents

to interpretation of the results and the planning of more effective work.

CONCLUSION

Teachers of mathematics have not been accustomed to consider the problems of testing from a broad point of view. Yet professional maturity, the welfare of the pupils, and the future of mathematics in the secondary schools all demand an ever deeper and broader study of instructional problems. In such study evaluation plays a fundamental role Curriculum revision, guidance of pupils, and improvement of instructional methods all depend upon careful evaluation. If it is superficial and narrow, the development of the place of mathematics in education is retarded If it is thoroughly and comprehensively done, there is much hope for the future.

CHAPTER X

THE EDUCATION OF TEACHERS

"When I say I'll learn a man . . . I mean it. And you can depend upon it. I'll learn him or kill him."

—MARK TWAIN, LIFE ON THE MISSISSIPPI RIVER

GENERAL CONSIDERATIONS

IN A consideration of general educational problems it is necessary to give prominence to the preparation of teachers. The part of the teacher is so important that other considerations, even very essential ones, are secondary by comparison. For instance, any insistence that is placed upon good objectives and good programs of study only emphasizes the importance of the teacher. Not until we turn our thought to the persons who are to carry on instruction do we reach the crux of the matter; for then it is that we pass beyond mere plans on paper to the personalities who will bring either success or failure to them.

There are two important postulates that underlie our efforts at mass education and that are related to the teacher. First, it is assumed that teachers can be secured in the large numbers required, and second, it is assumed that the necessary training will be given to them.

There is no reason to believe that the demand for teachers cannot be met numerically. It is true that the teacher's salary is often so small that it has been justly denounced as inadequate when account is taken of the social importance of his work. Yet it is sufficiently attractive for persons to seek it. There has been a relative stability to the teacher's pay, even in periods of depression, for economies that will make education suffer are likely to be applied with some moderation. The possibility that a posi-

tion may be permanent is certain to be regarded as something of a balance for the smallness of its wage. Herein, however, there lurks a real danger, for we should not be content to know that we can obtain teachers We need good teachers, and men and women who are truly able will be attracted into the profession in sufficient numbers only if there is adequacy as well as relative security of compensation

There are five major qualities that are to be considered in the mathematics teacher (*1*) social and civic attributes, (*2*) general culture, (*3*) familiarity with educational problems and theories, (*4*) skill in instruction, (*5*) knowledge of and interest in mathematics. It would be rather difficult to list the five attributes in their exact order of relative importance, but the Commission regards the last quality mentioned as distinctly the most important of all, after making an assumption, presently to be mentioned In the discussion that follows, (*3*) and (*4*) will be grouped under the general heading of *professional education*, a designation now very common

SOCIAL AND CIVIC ATTRIBUTES

Since it is undeniable that the personality and character of a teacher have much influence upon his success, an explanation is needed to justify considering knowledge of the subject taught as of more importance It is taken for granted that in the matter of personality and character the teacher is satisfactory. Persons who are clearly unsuited for teaching, because of poor personality, very poor health, or lack of sympathy for young people, should be eliminated early and they should not be allowed in practice teaching courses Although it is possible to overcome many personal handicaps through instruction and the exercise of vigilance, the instruction required may be of so individual a nature that universities and colleges cannot provide it. No greater responsibility rests upon officials and instructors who are connected with teacher education than that of preventing persons from entering the profession when they are obviously

unsuited for it. The course in practice teaching, which is now generally required, is especially adapted to serve as a safeguarding barrier.

One naturally expects that teachers will show civic interest, for the schools have a purpose beyond spreading the knowledge that is found in books and imparting the understanding that may be gained in the laboratory. Since an objective of the schools is to inculcate civic attitudes and ideals, a teacher should feel the responsibility of exemplifying some of the attitudes that the school says are important. Public problems should interest him, and he should be willing to bear a share of those burdens that are necessary for the common good. But neither civic nor non-curricular school activities should be expected or encouraged at the expense of good classroom work. The primary duty of the teacher is to instruct, and it is only a short-sighted policy that will put upon teachers demands that prevent a growth in scholarly attainment or general professional competence. It is to be expected that some teachers will have very discriminating tastes in the matter of using their leisure time. The greatest asset to any school system will always be those teachers whose time outside the classroom is spent in activities that are congenial to a person of intelligence and culture. In ways that may not be perceived by an undiscriminating observer they reach those boys and girls upon whom must rest the real hope of raising the standard and tone of American life. Such teachers should not be visited with official displeasure merely because they have no inclination toward certain popular activities

GENERAL CULTURE

The complaint is frequently made that knowledge is too often artificially divided into compartments. Of course such a situation as exists is to a large extent inevitable, being a consequence of the growth of civilization and a result of the progress that has been made in the arts, the sciences, and technology Each major field of learning has become so vast that it daunts even the most

industrious student, and if the education of the secondary school teacher of mathematics is to be adequate, a large part of his college work must be devoted to mathematics and allied subjects. But this fact is not an excuse for departing too much from the expectation that the teacher of high school mathematics should be a person with varied tastes and one who has interests outside of mathematics. Especially is emphasis to be placed upon knowledge of English and correctness in its use, both spoken and written.[1] One cannot expect to have good teaching of mathematics unless there is accuracy of expression, correct use of words, and a watchfulness of shades of meanings and the implications of sentences. The teacher of mathematics should be as careful in his speech and writing as the teacher of English, and he should help English instructors by the standards he requires in his own classes.

PROFESSIONAL EDUCATION

Professional education is older than is often realized By 1840 normal schools in America provided practice teaching, and in the early 1850's some universities did the same. In the 1874 catalogue of an academy one finds the statement: "Teaching is no longer an occupation, it is a profession; a profession which demands for its successful accomplishment as studied a preparation as that required for medicine, law, or the ministry." In the general program that was outlined we find "a course of professional instruction in which the science and art of Pedagogics will be presented with a discussion of all the minutiae of the school room work." The modern teacher may not be kindly disposed toward the word pedagogics, but it must be admitted that it has a candid directness. The Commission believes that professional training is an important part of the preparation of the teacher. We require far more teachers than can be obtained by selecting the "natural born teachers" Probably such indi-

[1] The present Commission endorses the statements made with regard to English by the National Committee of 1923 See its report, *The Reorganization of Mathematics in Secondary Education*, p. 84

viduals exist, but most successful teachers will admit that they have learned a great deal by experience. If this is so, it certainly should be possible to give valuable instruction and training to the future teacher while he is still a student. The Commission believes, however, that in America, the time spent in professional training has been carried in some cases beyond all reason, and it expresses the hope that members of education faculties who are conscious of this danger will take the lead in maintaining a sane balance.

The most important element in professional training is student practice teaching, carried out under the most competent supervision that can be procured. The Commission considers this work so important that it urges that even greater attention be paid to it in the future than in the past. Work in practice teaching should be preceded by a good course in methods; and in this course special stress should be put upon the topics that are most intimately connected with the ideas, the concepts, and the basic processes of mathematics. Some phases of the work of the methods teacher should be carefully correlated with instruction that students receive in mathematics.

Our public school teachers should certainly be acquainted with the history of American education; they should understand its philosophy, the problems that beset it, and the ideals it strives for. Incidentally, by way of broadening the viewpoint and giving the basis for a significant appraisal, they should know something about the schools of other countries. Such topics make up the content of the familiar course in secondary education. It would appear to be more in conformity with sound educational practice if a person were allowed to demonstrate his knowledge of them by an examination, without registered attendance at lectures in the subject. Not a great many prospective teachers would avail themselves of such a privilege, so that courses in secondary education would not dwindle away. But as the content of such courses is very simple, with little material that is in need of the exposition of a teacher, encouragement

should be given to the ambitious and industrious student who desires to follow a route other than the customary one.

The science of psychology has achieved an established place in teacher education. Since the teacher is dealing with human beings and is seeking to stimulate and arouse them to activity, he should know how to establish sympathetic connection with them. An understanding of other persons comes to some people quite naturally, and there are masters of the art of instruction who never sat in a class in psychology. But other teachers fail badly through ignorance of principles that they might readily have been taught. A course in psychology of moderate duration, devoted largely to its educational aspects and to the learning process, is an important general requirement. Teachers are generous in their praise of such a course, especially if they have had the good fortune to take it from an instructor who was himself a good teacher, and who by his own example could give conviction of the effectiveness of the principles that he set forth

The fact that this Report devotes an extensive chapter to the subject of testing is itself evidence of the importance that the Commission attaches to the problem. Systematic study should be devoted to it in the preparation of the teacher Study of educational measurement inevitably brings in statistical considerations, for which the prospective mathematics teacher should be well prepared by courses in his major department. It is not too much to expect that mathematics teachers in general should be helpful to colleagues in the subject of educational measurement and statistics

Professional preparation should not be terminated with the initial education of the teacher. In fact, one can doubt if a person can appreciate the importance of some aspects of such preparation without a year or more of active teaching experience. Formal courses, during graduate study in a university, may not be necessary, though they are valuable when real substance is put into them The teacher can grow professionally a great deal in the environment of his own school and through contact with

other teachers. Unless a school is so large that there are several teachers of a single subject, the only bond between them may be common educational problems. The only study that they can undertake together is study of general educational problems, and when this is well conducted by a good school administrator it may be as profitable as an additional educational course. The professional in-service growth of his teachers is certainly one of the major concerns of the principal and the superintendent.

TRAINING IN MATHEMATICS

The amount of study of mathematics that is required in order to obtain a license to teach the subject in a high school is far too low in most of the states. In some of them the prospective teacher need not have studied mathematics in college at all; so far as regulations are concerned, he is not obliged to have advanced beyond those subjects that he may be expected to teach. Even in states where there is a requirement of college study, the regulation can often be fulfilled by a year, or a little more than a year, of work. All of this indicates that however much we profess a devotion to education, and however zealous we are in providing buildings, equipping gymnasiums, transporting children to schools in rural communities, and doing other like things, we have not in any determined manner faced the essential problem of education, namely the competence of the teacher. By instituting certain professional requirements we have perhaps guarded against initial ineptitude in an instructor, and we have encouraged teachers to think about school problems; but in spite of that, we are content with the most modest attainments in the subjects they teach, at least so far as official requirements go. The situation suggests that although we desire something in this country definitely superior to bare literacy, we are not greatly annoyed by mediocrity Excellence or superiority does not stir us so deeply that we choose them as our watchwords and insist upon them in a grimly uncompromising manner. We desire that our boys and girls should be instructed

by teachers with social attitudes and community interests, but we do not demand that they have contact with teachers who are so well informed as to inspire, and who, having awakened interest, can assist their pupils toward superior attainment.

It is even argued at times that a scholarly teacher is likely to "talk over the heads" of his pupils, as if it is to be feared that a teacher may be too well informed. A teacher, especially an inexperienced one, sometimes does attempt too much and breaks connection with even the abler pupils, through forgetfulness of the needs of beginners. The danger may be particularly real in the case of mathematics, where understanding rather than mere memory is constantly involved. But a teacher who offends in such a way might still perform poorly if his knowledge of mathematics were even more limited. It is not to be forgotten that the verdict, "He knows his subject, but cannot present it," is often passed on teachers who really have only a modest knowledge when judged by a sufficient standard. The evil that is being considered is one that should be lessened by the practice teaching course, and experience in the classroom aids in its eradication. Administrators and supervisors should be only too glad to help correct such a defect when they are so fortunate as to have teachers who need help in bringing their daily classroom work *down* to the necessary level. It is far easier to make a correction of that sort in the case of a promising teacher than it is to bring the instruction of a teacher inadequately prepared in his subject up to the level to which the able pupils in his classes are entitled.

It is fortunately true that in many localities the preparation of teachers is distinctly above the minimum set in the regulations. A large proportion of teachers of mathematics in the larger schools today have had in college an undergraduate major in the subject, or at least a minor. But so long as defectively low regulations reflect education practices, a teacher prepared for some other subject may be found weakly and diffidently presiding over a class in algebra or nonchalantly listening to the dem-

onstration of a theorem in geometry. Pupils realize the impropriety of such occurrences, and we shall never create the respect for competence and knowledge until such instances become very rare. It is, of course, an excellent thing to have our practice actually better than our regulations, but the latter should be sufficiently high to command respect.

If mathematics were simply a tool subject, it still would be necessary to insist that teachers study branches of it that extend beyond the relatively small field to which the secondary school is restricted. It is true that a person may acquire a seeming proficiency with high school algebra by thorough study of a number of high school textbooks, and that he can gain a certain competence in geometric demonstration by painstaking perusal of the better and more exhaustive introductory books. But such narrowness of preparation is contrary to modern ideas of what education means. There is no margin of safety, no large surplus of knowledge and competence, always there to be relied upon. A teacher should not resemble an unhappy banker who, with reserves in a poor condition, nervously watches the door for the examiner. At any time the bright boy or the able and insatiably curious girl may cause considerable embarrassment. Nothing but study beyond the bare elements of a subject can make the elements seem simple and natural; nothing else can show their full significance.

A still stronger reason for requiring broad and thorough training on the part of the teacher grows from the cultural attributes of mathematics. The teacher's responsibility is not confined to the details of the lesson of the day, for he should seek to inspire and interest pupils in things superior to the contents of the book and the day by day discussion of the class. To pupils he should appear as a person with experience, knowledge, and capacity that far exceed the requirements of the moment. It should be manifest that he is interested in *thinking*, that he is a citizen of the world of ideas, of concepts, of mental enterprises, as well as a citizen of a political community with its vari-

ous social problems. Contacts with personalities of such a type should be among the life situations that schools give to boys and girls if we are to hold schools in esteem, and not be forced either to smile or weep over their shortcomings. Of course a large number of boys and girls, absorbed in adolescent impulses or engrossed in fancied responsibilities, as well as the exhilarating Huck Finns, will not be at all impressed by such teachers But this fact in no way affects the obligation of providing teachers with the characteristics described.

A detailed consideration of the contents of the college course of study is not necessary. In algebra and geometry, which are the basic subjects of the high school program, the study of the prospective teacher should carry him into fields that are definitely wider than he will need in his day by day teaching. His study of trigonometry should make him something of a master of that subject, so that the parts of it that find their way into the lower high school years will seem as simple as they truly are, while the parts that are taught in later years will present no difficulty to him. His contact with the calculus should be sufficiently prolonged and exacting to give a grasp of the principles and basic concepts that is so clear that it will last, and a grip of simpler techniques so firm as to be durable. The calculus being in a way the beginning of modern mathematics, this Commission believes that it is as bad for a high school teacher of mathematics not to have some clear conception of it two hundred and fifty years after its beginning as it would be for him not to know where London and Paris are, or to be unable to tell something of the story of Benjamin Franklin. The teacher should certainly be able to give an intelligent answer to the curious boy or girl who may ask a serious question as to the nature of the subjects studied in the calculus. It is to be recalled, furthermore, that parts of elementary differential calculus have been suggested in earlier chapters of this Report as topics for the last year of the high school. A study of mathematical statistics and of the mathematical theory of finance is

also important in addition to the study of the traditional subjects.

The Commission believes that it should stress one subject, namely the history of mathematics. If the study of secondary mathematics is to reveal mathematics as one of the fundamental enterprises of man, which, though rooted in daily need, is an expression of deep, irrepressible, and idealistic impulses, then the teaching of it should constantly be associated with its history. One recalls the statement of Glashier, "I am sure that no subject loses more than mathematics by any attempt to dissociate it from its history."[2] The history of mathematics, however, is too often treated by our universities and colleges as a stepchild. Though courses in it are provided, the Commission fears that too often they are dry and sterile, and it believes that they will remain so until the subject is properly appreciated and steps are taken to develop more scholars who are ardent in the cultivation of the history of mathematics and enthusiastic in its teaching.

Attention has already been called to the increasing number of subjects that are becoming mathematical in nature. The mathematics teacher may have acquaintance with this general fact, and know a little as to the nature of the applications, without having carried on systematic study of the related subjects It is, however, quite desirable for him to have had a good college course in physics, or chemistry, or astronomy, of such a nature that the employment of some mathematics was needed. As a high school teaching subject to be combined with mathematics, physics continues, as it has long been, a very natural related subject for many teachers of mathematics.

But another consideration should be touched upon, even if slightly. There are many mathematics teachers who have preferred foreign language, frequently Latin, as a minor, and others who have been drawn toward English, or toward history. The

[2] Glashier, J W L, "Mathematics and Physics," Presidential address before the British Association for the Advancement of Science, Section A, 1890 *Nature*, Vol XLII, p 466.

Commission regards such combinations as having much merit, and would not discourage them. However correct may be the assertion that language study does not employ mathematics, the question involved is not disposed of by so obvious an assertion Many investigations have shown a striking correlation between grades in mathematics and grades in foreign language. Both studies are definitely intellectual, with a good firm texture unalloyed by the trivial and the commonplace, and it is not strange that some persons should desire to teach two subjects that spring from congenial impulses. Many strong students of mathematics are not attracted to the laboratory with all its paraphernalia, and it is doubtful if there is a gain when, for a second teaching subject, we force them away from a natural inclination into a study that has unattractive aspects, just because the latter uses formulas and employs equations. It is also to be recalled that, if our schools are to exemplify culture in a broad sense, it is a wholesome thing to have some teachers cut across traditional lines and reveal an interest in subjects that on the surface seem unrelated The teacher of mathematics who is competent to teach Latin and enjoys doing so, should not feel obligated to explain why he does not prefer meter sticks and galvanometers to *amo, amas, amat.*

It should be clear that the teaching of English may well be congenial to some teachers of mathematics, both on account of the part that deals with grammar and the part that deals with literature. In fact, if the great importance of English training to the mathematics teacher is stressed, we are quite certain to find ourselves producing some teachers competent in the two fields. As to the teaching of history, let the following remarks suffice. There is a tendency to have history deal more with the development of culture and less with political and military matters. The opportunity for the teacher of mathematics is obvious, especially if he has given thorough study to the history of science

Although the traditional "major work" of the university or

college department of mathematics has been for the most part quite well conceived so far as content is concerned, its actual bearing on secondary education has too often been left for the student to infer. Moreover, university and college teachers have not always kept in touch with the problems of secondary education, even when a large number of their more advanced students were preparing for high school positions. Every mathematics department of any size should have some member, well trained in mathematics, whose dominant concern is the teaching of mathematics in the high school. In addition to an awareness of the difficulties of the problems of secondary education, such a person should reveal a strong devotion to the subject that he teaches Though he may not be engaged in technical research, there are plenty of ways in which he can show an extensive knowledge, activity, and enterprise.

There is an increasing number of teachers of mathematics who take graduate study, and in many of the larger high schools teachers are expected if not required to hold the Master's degree. Though some states now require that all high school teachers hold the degree, it is to be observed that such a prescription may not be accompanied by a comparable increase in training in the subjects that the teacher teaches, strange as this may seem to many persons. It is also to be observed that if the output of persons receiving a Doctor's degree should exceed the demand of colleges and universities—a situation that seems likely to arise—many of the recipients of such degrees will naturally find positions in junior colleges and the larger high schools.

Several very important questions arise in connection with graduate work in mathematics for high school teachers. Although the traditional training has been quite suitable for those teachers who are strongest mathematically, it can be doubted whether it is best suited for many others Certainly there is a great wealth of good mathematical material from which courses could be built that would be appropriate for different types of high school teachers, although a too diversified list of offerings

is not to be expected; and the graduate programs of universities should not be devised solely for those persons who expect to take the traditional degrees in the expectation of teaching in a university or a college. A commission of the Mathematical Association of America has recently published[3] a "Report on the Training of Teachers of Mathematics," which considers various aspects of the problems here briefly alluded to.

SPECIFIC PROGRAMS

There is given below an outline of the training for mathematics teachers that the present commission recommends, the recommendations being in essential agreement with those of the commission of the Mathematical Association, to which reference has just been made.[4]

HIGH SCHOOL TEACHERS OF MATHEMATICS AND A SECOND SUBJECT

A large number of teachers are employed in high schools in which they must teach more than one subject, and in some states where certification in specific subjects is required, a candidate must qualify in two subjects. A committee of the North Central Association of Colleges and Secondary Schools has recently recommended[5] that learned societies, "in order to protect their own immediate interests as well as to improve the American high school . . . should undertake the task of defining the make-up of the broad fields within which their own subjects should be included." It further states, "It is clear, for example, that physics and mathematics are inseparably related

[3] *American Mathematical Monthly*, Vol XLII, pp. 263-277, 1935

[4] In its tabulated recommendations the former commission listed the study of English. Although this subject is not mentioned in the tabulation of the present report, enough has been said already to show that the present commission endorses fully the statement of the former commission with regard to English, as well as its statements about language, literature, and other cultural studies

[5] See the report "Inadequacies in the Subject Matter Preparation of Secondary School Teachers and Suggestions for Their Corrections," *North Central Association Quarterly*, Vol XII, pp. 439-539, 1938

and together might be regarded as a sufficiently broad field." There is no question at all as to the great breadth of the field of learning composed of the physics and the mathematics of today, but it is to be noted that there is a non-reciprocal character in the relation of the two subjects that gives rise to a grave question if one seeks to make a permanent union of them. Mathematics is necessary for most parts of physics, unless the latter is made purely descriptive, but physics is not fundamentally essential for most of mathematics. The present Commission has already called attention to the fact that many good students of mathematics, finding no special appeal in laboratory manipulation, would prefer a second teaching subject outside the broad field recommended above. Especially in the case of women there is likely to be an attraction toward a foreign language, toward English, or toward history; and the Commission does not believe that the necessity of preparing a teacher for two subjects prevents a non-mathematical subject from being joined successfully with mathematics. The teacher will presumably have taken a major in one of the subjects he teaches, and a minor in the other, though in some current teacher training programs, relatively equal weight is given to each of two teaching subjects. As a rule, a teacher of mathematics and a second subject will not be giving the more advanced courses in mathematics, such as were outlined for the twelfth year in earlier chapters of this Report Consequently a somewhat more restricted training in mathematics might be accepted than in the case of a teacher in a school where more advanced courses are offered

The teacher's training should include:

(*1*) In mathematics.

(*a*) Courses including complete treatments of college algebra, analytic geometry (including a little solid analytics), and six semester-hours of calculus

(*b*) A course that examines somewhat critically Euclidean geometry, and gives brief introductions to projective geometry and non-Euclidean geometry, using synthetic methods (three semester-hours).

(*c*) Advanced algebra, including work in theory of equations, mathematics of finance, and statistics (six semester-hours). This course should give some careful attention to the basic laws of algebra, to the nature of irrational and complex numbers, and operations with them. It should be throughout somewhat critical and not purely manipulative

(*d*) Either directed reading or a formal course in the history of mathematics and its concepts.

(2) In related fields.[6]
An introductory course in physics, astronomy, or chemistry that makes use of some mathematics.

(3) In professional preparation.[7]

(*a*) A course in methods (two or three semester-hours). This work should be given by a person who has had a good mathematical education and also experience in high school teaching

(*b*) A course in secondary education (three semester-hours). Some consideration of educational philosophy and of the history of education can be given in this course

(*c*) A course in psychology, with emphasis on its educational bearing and on the problem of learning (three semester-hours).

(*d*) A course in educational tests and measurements that employs some statistical material (two semester-hours).

(*e*) Practice teaching. It is not usually possible for a student to have practice teaching in two fields. If mathematics is his major he should have practice teaching in that subject.

HIGH SCHOOL TEACHERS OF MATHEMATICS ALONE

A teacher whose full time is devoted to mathematics and who is in a school that provides courses in more advanced secondary mathematics should have, in addition to the training previously described, as much as possible of the following work

(*1*) Advanced calculus and differential equations or mechanics (six semester-hours).

(*2*) Additional work in geometry, such as projective geometry, descriptive geometry, etc. (three semester-hours).

[6] The related subject is not regarded here as a teaching subject.

[7] The courses mentioned are likely to be included in state regulations, and it is taken for granted that a prospective teacher actually prepares to meet the professional requirements of the state in which he wishes to teach

(3) Additional work in algebra, including some modern algebra (three semester-hours).
(4) At least one more of the three sciences of physics, chemistry, and astronomy.

JUNIOR COLLEGE TEACHERS

The teacher of mathematics in a junior college should have at least the Master's degree in mathematics, and should preferably have some graduate work beyond that normally included in preparation for that degree. Irrespective of whether a legal requirement exists as to professional preparation, he should have given attention to this question. He should be a close student of educational problems, especially those that affect the junior college, so that he can assist in directing this new enterprise in American education along sound lines. He should be able to institute and carry on successfully instruction for the commercial and vocational groups mentioned in Chapter VIII as well as for the academic group.

APPENDIX

APPENDIX I

ANALYSIS OF MATHEMATICAL NEEDS

"The faculty for truth is recognized as a power of distinguishing and fixing delicate and fugitive detail."

—WALTER PATER, COLERIDGE

THERE will be given here an analysis of the mathematical equipment needed for the activities and experiences of life. In discussing this question, the word *need* will be understood to denote not only such knowledge or capacities as may be indispensable, but also attainments that may profitably be used in either a utilitarian or a cultural manner. In a very real sense such knowledge and capacities are actual needs, to be provided for by the schools. For it must be remembered, as has been said before, that the school has an obligation to create capacities of one kind or another, and should explain to pupils the advantages which may result from them, though it recognizes that in many cases the capacities will not all be employed. For reasons that seem obvious, the needs will be discussed under three headings: "For ordinary life"; "For leadership and higher culture", "For specialized use as a vocational tool."

FOR ORDINARY LIFE

Arithmetic. The paramount mathematical need of the average citizen is for a greater knowledge of arithmetic than is now common. By arithmetic is meant more than computational facility and understanding of principles There is needed also familiarity with applications to a wide variety of problems or situations that confront people, and ability to understand certain mathematical ideas and procedures that may be encountered in ordinary reading.

From among the many activities of ordinary life which require some use of arithmetic, a few typical examples are listed here:

(1) *The home:*

(a) Budgeting income, keeping accounts, checking bills.
(b) Quantity and installment buying.
(c) Estimating depreciation on home or car.
(d) Buying or settling fire or burglary insurance

(2) *Personal finances*
   (*a*) Handling funds—depositing, checking, remitting.
   (*b*) Borrowing, paying loans and interest.
   (*c*) Saving and investing, choosing securities.
   (*d*) Buying life and disability insurance, and annuities.
   (*e*) Paying taxes, contributing to charities.

(3) *Recreational activities*
   (*a*) Buying season tickets for sports, plays, lectures, music.
   (*b*) Planning trips—expenses, time schedules.
   (*c*) Arranging social functions.
   (*d*) Helping with neighborhood entertainments.

Among the numerous items often encountered in general reading matter, the comprehension of which requires a considerable knowledge of arithmetic, may be mentioned some relating to.

(1) *Civic and social life:*
   (*a*) Taxes—property, income, inheritance, sales, imports.
   (*b*) Use of public funds—for roads, public defense, social security, health, interest on debt.
   (*c*) Social statistics—population, vital statistics, distribution of wealth, dependency.

(2) *Economic conditions:*
   (*a*) Industrial activity, volume of output, dividends.
   (*b*) Agricultural acreage, crops, prices.
   (*c*) Labor conditions, wage levels, unemployment.
   (*d*) Price indices, real wages, living standards.
   (*e*) Financial conditions, interest rates, public debt, monetary standards.
   (*f*) Exports, imports, international exchange.

(3) *General information:*
   (*a*) Scientific and technical advances, involving use of very large or very small numbers; e.g., relating to stellar distances, infra-molecular physics, engineering feats, number theory.
   (*b*) Statistical data involving averages—mean daily temperature, or average annual rainfall; average crop per acre; average speeds
   (*c*) Further data involving ratio, proportion, or percentage, dietetic and budgetary matters, probability of death or illness, "expectation" in games of chance, cost of essential elements in materials containing large portions of waste.
   (*d*) Diagrams, records, puzzles in reading material for recreations, and in accounts of sports.

Graphic Representation. To understand numerous mathematical

ideas which are commonly encountered in general reading, one should be able to interpret graphs such as the following:

(*1*) *Line graphs:*
As used for market quotations, growth of population, progress of a campaign, course of a fever temperature.

(*2*) *Bar graphs:*
Showing the distribution of telephones, radios, automobiles.

(*3*) *Circle graphs:*
Portraying the percentages of land given to cultivation, grazing, forest, waste; or the distribution of the national income.

(*4*) *Scale drawings:*
As used in surveying, or to find forces or velocities.

Algebra. Although the average citizen rarely or never uses formal algebra as an instrument of calculation, it could so be used occasionally to advantage if its possibilities were understood. For instance, in studying investments, installment buying, statistical aspects of one's business, solving mixture problems, or indeed in dealing most effectively with various matters listed under arithmetic above, one needs some command of formal algebra. For the average citizen, however, the chief need is for such algebraic knowledge as will give him insight into and appreciation of mathematics aspects of the modern world, scientific achievements, economic questions, and the like. In particular he needs to be familiar with:

(*1*) *Positive and negative numbers:*
Their frequent use in describing such opposites as assets and liabilities, forces or motions in opposite directions, temperatures above and below zero, rates of increase and decrease

(*2*) *Formulas and the function idea:*
Such related variables as the age and value of a house, amount of use and annual cost of a car, driving speed and distance required for stopping, length and period of a pendulum.

(*3*) *Equations:*
Their constant employment in technical and scientific work, as a general way of finding unknown quantities.

(*4*) *Coordinates:*
Their more common use in plotting curves, mapping, and as a basis for engineering projects.

(*5*) *Indirect measurement:*
Finding elevations of inaccessible peaks, or constructing a long tunnel from both ends, understandable on the basis of numerical trigonometry using simple algebraic equations.

(*6*) *Modern methods of calculation.*
The power of generalized exponents as logarithms.

(7) *Permutations and combinations:*

Their frequent application in devising distinctive labels, etc.

Informal Geometry. Some geometric knowledge is occasionally employed as a working tool about the home, in such activities as:

(1) *Making mensurational calculations.*

(2) *Using a protractor and other instruments.*

But, as in the case of algebra, the most important uses that the ordinary citizen can make of geometry are those by which he may achieve insight. For instance, a good foundation in geometry will contribute materially toward an informed appreciation of matters relating to:

(1) *Architecture and decorative art:*

Recognition of the skillful use of geometric relations in patterns, arches, vaults, columns, buttresses, apses, windows.

(2) *Engineering and manufacture:*

Realization of the importance of triangles for rigidity of structures; of angles in surveying, geography, and navigation; of circles in machinery and appliances; of polygons in furniture, jewelry, diamonds.

(3) *Natural forms:*

Awareness of geometric figures in natural objects, terrestrial, and celestial, eclipse phenomena, stellar configurations, hexagonal cells, crystals.

By way of summary it is to be said that the important uses of mathematics for the ordinary citizen are in large measure cultural. Mathematics provides an outlook, and a means of understanding. There are important aspects of the world that only mathematics can interpret to the citizen. Mathematics furnishes a mode of thinking about many aspects of life, and a very general kind of language. A liberal view of education regards such matters as genuine needs of even the ordinary citizen. It rejects the thought of a person being satisfied with such a minimum working equipment as would enable him to exist as nothing more than a "hewer of wood and drawer of water."

FOR LEADERSHIP AND HIGHER CULTURE

Next will be considered the further mathematical equipment which is important for most persons who are to occupy positions of responsibility and leadership, or who seek through higher education to attain a broad culture marked by a catholicity of interest.

Public Service and Leadership. Individuals who exercise large influence on either organizations or communities should understand public affairs thoroughly. Some mathematical aspects of such a many-sided competence are these abilities:

(*1*) *To judge the significance of complicated numerical data:*
Here one needs a knowledge of statistical analysis, involving considerable mathematics.

(*2*) *To follow quantitative studies of social phenomena where necessary:*
Studies of cyclic changes commonly employ trigonometric analysis, those dealing with population growth and other trends often involve calculus.

(*3*) *To recognize fallacious conclusions:*
- (*a*) By checking against known facts.
- (*b*) By scrutinizing critically the authority for each step in the argument and the cogency of each implication.
- (*c*) By realizing that the validity of conclusions depends not only on correct steps of deduction but also on the basic assumptions or presuppositions

Cultural Satisfaction. The more highly educated group of persons who wish to be familiar with the most significant modern theories, the central problems, and the principal methods of investigation, in various fields, can make good use of mathematical knowledge far beyond the elementary field:

(*1*) *To understand quantitative procedures:*
Through some familiarity with calculus and with group concepts

(*2*) *To follow philosophical discussions:*
- (*a*) As to the nature of space. using ideas of non-Euclidean geometry.
- (*b*) As to the nature of number: complex numbers, series, theory of classes.
- (*c*) As to the problem of knowledge: foundations of mathematics
- (*d*) As to the nature of scientific law: functions of a real variable, and of many variables; point sets.

Obviously laymen can scarcely devote to mathematics all the time needed for high achievement along all the lines mentioned; but, to the extent to which the indicated equipment is lacking, in so far are an individual's potentialities circumscribed

FOR SPECIALIZED USE

The mathematical needs of those who are to use this subject as a working tool in their vocations or professions differ greatly, according to the field.

The Physical Sciences. Here the needs are especially extensive. There is scarcely any limit to the amount or variety of mathematics

that can profitably be employed in some branches of physics or in certain laboratory researches. Although laboratory work in chemistry is not particularly mathematical, it is impossible to follow work in physical chemistry or chemical theory without a substantial knowledge of higher mathematics. The following equipment is in very common use:

(1) *By physicists:*
Advanced calculus, probability, theory of functions, differential equations, Bessel functions, Fourier and related series, calculus of variations, differential geometry, vector analysis, tensor analysis, matrices, group theory, and so on.

(2) *By chemists:*
Calculus, including advanced topics, differential equations, group concepts.

(3) *By pharmacists:*
Percentage, proportion, formulation and solution of simultaneous linear equations relating to mixtures.

Engineering and Related Activities. Handbooks and shop rules embody results of mathematical work, and render unnecessary the direct use of calculus and higher subjects in routine work. To meet situations which are new, engineers must have a knowledge of the mathematical principles involved, and must sometimes formulate and solve differential equations. The mathematical requirements of engineering schools are doubtless reliable indications of a suitable equipment in this field:

(1) *For engineers.*
At least calculus and differential equations, besides analytical geometry and descriptive geometry

(2) *For engineering employees.*
As a minimum, facility in handling algebraic formulas and graphs, and mechanical drawing, a good knowledge of geometry; some familiarity with trigonometric functions and the solution of triangles. Larger equipment is very helpful.

(3) *For skilled mechanics:*
Ability to apply arithmetic principles, to compute without making mistakes, to determine the degree of precision required in instruments, and to keep calculations correspondingly accurate; facility in dealing with formulas and graphs that involve positive and negative numbers, square roots, quadratic and simultaneous equations, sines, logarithms, very large and very small numbers, occasional interpolation to seconds, knowledge of many geometric facts needed in working with triangles, and experience in analysis, an extensive mathe-

matical vocabulary and ability to read technical books and articles.

The Earth Sciences. In physiography, meteorology, geology, astronomy, and such related fields as cartography, geodetic surveying, navigation, and aeronautics, the situation as to mathematical needs resembles that in chemistry. Many branches of the work are not particularly mathematical; others are highly so. Essential equipment is somewhat as follows:

(1) *Physiography and cartography:*
A good command of trigonometry, some analytic geometry and calculus.

(2) *Meteorology:*
Elementary mathematics through integral calculus.

(3) *Geology:*
(a) Locating oil deposits: graphic, algebraic, and geometric techniques.
(b) Stratigraphic work: trigonometry
(c) Seismology. elementary mathematics through integral calculus.

(4) *Navigation* (marine or aerial):
Trigonometry at least

(5) *Ballistics:*
Analytic geometry, calculus, differential equations.

(6) *Practical astronomy:*
Elementary mathematics including spherical trigonometry.

The Life Sciences. Most of the work here is not mathematical; but there are divisions in which certain algebraic, geometric, trigonometric, or statistical techniques are employed, and some requiring calculus and differential equations. In listing the following typical examples of mathematical aspects of biology and related fields, elementary algebraic, graphic, and statistical methods will not be explicitly mentioned since they occur so generally

(1) *Biochemistry:*
Production of lactic acid by bacteria
Coagulation of blood by adrenalin.
Rate of digestion as related to the amount of pepsin present—this study involving a differential equation.

(2) *Other metabolic studies.*
Healing of wounds—the celebrated work of Carrel and Du Nouy.
Pathology of bones and other parts of the body
Production of red blood corpuscles
Respiration and nutrition

Metabolic rate as a function of body weight in mammals. (All these studies use involved irrational algebraic functions, and some employ nomographic charts.)

(3) *Light biology.*
Depth of penetration of tissues by light of various intensities
X-ray therapy (use of semi logarithmic charts).
Formation of chemical products in the retina for different intensities of light.
Fluctuations of luminous intensity in certain organisms.

(4) *Heat and energy problems:*
Daily expenditure of energy at various ages.
Effects of different types of clothing
Mechanical efficiency of muscles during work.
Circulation of the blood (use of hydrodynamic principles).

(5) *Growth and senescence:*
Growth curves—studied by T. B. Robertson and others, using differential equations.
Growth and form—studied by D'Arcy Thompson and Julian Huxley.
Tissue culture; regeneration of tadpole's tail; etc.
Effects of diet and light upon growth.
Senescence of various types—studied by James Gray and others.

(6) *Population problems:*
Census Bureau studies, using multiple integrals.
Human and insect populations—studied by Raymond Pearl, using differential equations.
Colonies of bacteria, studied by using calculus
Actuarial problems, requiring probability, calculus, and higher analysis.

(7) *Genetics:*
Mendelian and other questions of heredity—using algebra and probability.
Genetic theories of natural selection (R. A. Fisher); mathematical theories of evolution (G. U. Yule).

(8) *Forestry and agricultural experimentation:*
Using sampling theory and other mathematical analysis.

(9) *Biometry and anatomy:*
Studies of arterial form involving trigonometry.
Studies of measurements and ratios (some used in medicine), involving frequency curves and correlation.
Relation of body area to weight, involving nomographic charts and Du Bois' formulas

(10) *Neurology:*
Electromotive force generated by stimulating a nerve.

Current strength needed to excite nerves of different lengths.
Changes in velocity of propagation of a nerve impulse.
Reaction times in plants and animals for varying stimuli.
Effects of narcotics.

Psychology and Education. Besides the extensive use of statistical methods in psychology and education, and the use of matrices in some of the newer studies, such as those of L. L. Thurstone in his *Vectors of Mind*, the following mathematical sections of the field may be mentioned.

(1) *Threshold stimuli and the Weber-Fechner law:*
Some use is made of calculus.

(2) *Problems of vision:*
Campimetry, using trigonometry and polar coordinates.
Helmholtz's studies, using integral calculus.
Psychometrics—with its phi gamma curves, and differential equations.

The Social Sciences. In the social sciences, economics makes the largest demands upon mathematics, though political science and sociology make considerable use of statistical methods, and employ growth curves in connection with cultural change and the development of institutions. Subdivisions of economics which utilize mathematics beyond arithmetic, including calculus in some cases, are in part the following:[1]

(1) *Synthetic economics* (including statistical laws of demand and supply).
(2) *Economic trends and cycles.*
(3) *Index numbers* (and the purchasing power of money).
(4) *General economic theory* (including equilibrium, monopoly, effects of taxation, and related matters).
(5) *Mathematics of finance and accounting.*

Industry and Commerce. Here the mathematical tools most frequently needed are statistical in character: analyses for the information of directors; and sampling theory for the control of output. The mathematics of finance is also highly important here

Philosophy. What was said on page 211 as to the needs of the educated layman who wishes to follow philosophical discussions applies here with even greater force to the professional philosopher, for whom a knowledge of symbolic logic is also becoming important.

[1] Instead of going into detail here reference is made to the report of a committee of the Social Science Research Council on "The Collegiate Mathematics Needed in the Social Sciences," where an extended discussion may be found, *American Mathematical Monthly*, December, 1932, pp. 569-576 See also H Hotelling, "Some Little Known Applications of Mathematics," *The Mathematics Teacher*, April, 1936, pp. 159-169

Aesthetics and the Fine Arts. A broad familiarity with the principles of geometry will be found helpful in this field. This is obvious as regards architecture and some decorative arts; and is doubtless true in reading such works as Ruskin's on "Drawing and Design" Efforts have been made to determine a mathematical measure for aesthetic qualities.

In the field of literary criticism the work of W. W. Gregg, *Calculus of Variants* shows what may in the future be deemed a need of some scholars in that domain.

APPENDIX II

THE TRANSFER OF TRAINING

"Excited as he was, he could have had no conception of the adventures to which his mathematical calculations were destined to lure men."

—JOHN NOEL, THE STORY OF EVEREST

THE Commission does not deem it necessary or desirable to present an elaborate study on the Transfer of Training. As the problem, however, has had an important place in recent educational thought, a brief review will be given of the main contributions and suggestions that have emerged from recent studies and that seem to be especially significant for the present Report.

The importance of the attitude of teachers toward the question of Transfer of Training can hardly be overemphasized. According to Whipple, it has been for many years "acknowledged to be the central problem of educational psychology."

The doctrine of "formal discipline" as a theory of learning, held many years ago, furnished a valid reason for the inclusion of mathematics in the educational program. This theory was questioned by many psychologists and as a result of investigation was given up as a theory of learning A new theory, known as Transfer of Training, emerged. Transfer of Training as defined by Orata is "that process of using or applying previously acquired information, habit, skill, attitude; or ideals in dealing with a relatively new or novel situation."[1] This theory has been the subject of scores of investigations, and even now is central in the study of the educational process A large part of the experimental investigation was claimed to invalidate the theory, and as a result, mathematics rapidly lost its strong position as a required subject in the high schools

The Commission feels that it is desirable to point out that most of this experimental approach was in no sense scientific, and never should have been treated as such. Prior to 1916, out of twenty-five important studies the subjects of investigation in eighteen cases were graduate students and instructors of psychology Very important

[1] Orata, P T "Transfer of Training and Reconstruction of Experience." *The Mathematics Teacher,* Vol XXX, pp. 99-109, 1937

conclusions were reached even though no one claims that the learning processes of children are the same as those of adults. In the famous Thorndike-Woodworth experiment of 1901 a single group of adults was used. In many experiments fewer than seven subjects were used.

Many psychologists and educators have protested against the non scientific investigations that were made the basis for far-reaching changes. Hollingworth's criticism is typical of that leveled against this type of study. Referring to the controversy of transfer he says, "It is a long and pitiful story of the blind leading the blind. . . . In due time the experimental method was applied, and a long series of studies appeared, for the most part inadequately planned and often erroneously interpreted."[2] Bagley speaks even more directly, "As a matter of plain fact, the evidence was (quite without deliberate intent) actually distorted. Some of the professors of education who followed carefully the published reports of the 'transfer' experiments as they appeared recognized this and protested against it. Many professors of education, however, took their evidence, not from the primary sources, but from interpretations of the evidence from the point of view of a narrowly mechanistic psychology."[3]

In the famous Thorndike experiment of 1924, in which an effort was made to discover the disciplinary values of the high school studies, it was reported by Thorndike: "The expectation of any large differences in general improvement of the mind from one study rather than another seems doomed to disappointment."[4] It was revealed that one might as well study sewing as mathematics so far as any development of the thinking habits were concerned. However, near the close of the report one reads, "A repetition of this experiment with 16,000 or 18,000 more cases is needed before final conclusions should be stated." Most educators who were planning for the great mass of incoming students probably never read that far in the report.

Most of the early experiments on transfer dealt with the mechanistic or specific habit types of learning. The idea underlying these investigations was that "the mind is not a collection of general powers or functions, such as observation, attention, memory, reasoning, and the like but that it is the sum total of countless particular capacities."[5] A fundamental weakness of this philosophy is that,

[2] Hollingworth, H L *Educational Psychology*, pp 404, 406 D Appleton Century Co, New York, 1933

[3] Bagley, W C *Education and Emergent Man*, p 87 T Nelson & Sons, New York, 1937

[4] Thorndike, E L "Mental Discipline in High School Studies." *Journal of Educational Psychology*, Vol XV, pp 97 98, 1924

[5] Orata, P T. *Theory of Identical Elements*, p 5 Ohio State University Press, Columbus, 1828

from processes involving rote learning, mechanical habits and drill, which have little to do with real learning, no transfer would ever be expected to result. If the teaching and experiments had been planned so that large concepts and broad generalizations had been taught and tested, the results would have been different, as indeed they were in later, better conceived efforts. The following clear statement as to confusing lower levels of learning with higher understanding has been made by Dewey:[6]

"In some educational dogmas and practices, the very idea of training mind seems to be hopelessly confused with that of drill which hardly touches mind at all—or touches it for the worse—since it is wholly taken up with training skill in external execution. This method reduces the 'training' of human beings to the level of animal training. Practical skill, modes of effective technique, can be intelligently, non-mechanically used, only when intelligence has played a part in their acquisition."

In the 1922 Thorndike study, ninety-seven graduate students were asked to square $x + y$. There was an error of 6% of the results. Then the group was asked to square $a_1 + a_2$, with a resulting error of 28% of the results. According to the specific habit psychology here are two distinct learning situations. There was no transfer. Any teacher of mathematics knows that there is one learning situation here that is a generalization applicable to the squaring of all binomials It is very doubtful if any real learning situation had been developed and hence there could be no transfer.

Another school of psychology interprets learning as a result of a generalization of experiences. This demands understanding, a development of concepts, a conscious generalization of many specific related experiences Thus, if the processes of arithmetic are learned, algebra, except for the notation involved, offers far less difficulty than when it is considered as distinct from arithmetic

Professor Judd, who has for many years advocated the importance of training in generalizing as a way of promoting transfer, has given considerable attention to the case of transfer in mathematics. Here much depends upon how the subject is handled, and Judd says, "The most promising subject in the curriculum can be turned into a formal and intellectually stagnant drill if it is presented by a teacher who has no breadth of outlook and no desire to teach pupils how to generalize experience. On the other hand, a teacher who has the ability to train pupils to look beyond particular facts and to see their relations and their broader meanings can stimulate thinking with any material of instruction that comes to his hands."[7]

[6] Dewey, John *How We Think*, p 52 D C Heath and Co, 1933

[7] Judd, C. H. *Psychology of Secondary Education*, p 422 Ginn and Co., 1927.

In recent years many experiments have been conducted in which conscious attempts have been made to produce transfer. Along with these experiments others have been set up with no effort being made to secure transfer. In one of the best studies in transfer, "The Theory of Identical Elements," by Orata, a summary of these experiments is given In the first group of experiments, transfer was very evident. Orata makes this observation:

"It is evident from this experiment that the automatic transfer of training from Latin to English is comparatively small; and that it can be increased three-fold or four-fold by proper selection and organization of subject-matter and proper methods of teaching"[8]

The Commission could offer much supporting evidence to the thesis that if we wish to transfer we must teach for it It is the old saying in different language, "We reap only what we sow." A very helpful suggestion comes from another representative of the field of education, Dr. Freeman.[9]

"Whether education improves the intelligence will depend largely on whether the education is of a narrow specific sort or of the broader sort which encourages the development of ideas and generalizations. The kind of education we give the child furthermore will depend upon whether we believe in the possibility of the broader outcomes of education. This being the case it seems wise to give the child the benefit of the doubt and to provide for him the type of education which will develop his intelligence to the highest point of which he seems capable."

The education of young people must be a cooperative effort. No subject will transfer automatically Very early in the experimentation on transfer an effort was made to find out if a habit, such as neatness, would carry over to fields not included in the experiment The teacher of arithmetic was to use all efforts possible to secure neatness in the class in arithmetic. The improvement in arithmetic was very noticeable, but the same pupils showed not the slightest improvement in language and spelling papers. In a later experiment neatness not only as a habit but as an ideal was held up in not only a particular class, but in dress, appearance, business, in fact everywhere except in other classes Three months were required for the experiment, which included pupils of grade seven in three cities W C Ruediger who conducted the experiment reported· "Evidently neatness made conscious as an ideal or aim in connection with only one school function does function in other subjects."[10]

[8] Orata, P T, *op cit.*, p. 139

[9] Freeman, F. N. "The Effect of Environment on Intelligence." *School and Society*, Vol. XXXI, pp 623 632, 1930

[10] Ruediger, W C "Improvement of Mental Functions through Ideals" *Educational Review*, Vol XXXVI, pp 364-371, 1908

In an article by W. O. Brownell, giving a summary of transfer studies recently made, one reads: "The possibility and the desirability of transfer cannot be questioned. The problem thus becomes one of so organizing the materials and methods of instruction so guarantee the largest amount of positive transfer."[11]

Transfer, when detached from the efforts of the learner, is a problem that can be solved only by the joint efforts of all teachers and by properly uniting various forces that contribute to the development of children. If the teaching of the larger outcomes of learning is not secured, no positive transfer can be expected. Quoting again from Orata in the recent article:

"The various subjects of the curriculum, most of all mathematics, cannot be expected to result in automatic transfer to the social situation that confronts the child, unless by proper instruction and organization of the school, these subjects can be made a 'way of life' and be so regarded and used by the child himself. This is what we mean by 'humanizing education in the concrete.' It is the kind of teaching that will promote intelligent and functional learning."[12]

The question raised by the quotation from Orata is that of transfer of the results of mathematical study to social situations and to specific problems that confront persons throughout life. Such a transfer is certainly an important concern, and a course of mathematical study should be planned so as to make it possible, and teachers should teach in such a way as to achieve that goal. The thought that the transfer that is so earnestly desired may not be automatic is one to be kept constantly in mind.

But there is another aspect of the broad question of transfer that extends it quite beyond its purely social phases. It is described as follows by Judd.

"At the higher levels transfer is typical, not exceptional. Indeed, the function of the higher mental processes is to release the mind from particulars and to create a world of general ideas Thus, when the intellectual efforts of the race evolved a number system, it became possible to deal readily with every situation in which quantity is involved; when languages were developed, men found themselves in possession of means of communicating on every conceivable topic. The psychology which concludes that transfer is uncommon or of slight degree is the psychology of animal consciousness, the psychology of particular experiences. The psychology of the higher mental processes teaches that the end and goal of all education is the development of systems of ideas which can be carried over from the

[11] Brownell, W A "Theoretical Aspects of Learning and Transfer of Training" *Review of Educational Research,* Vol VI, pp 281-290, 1936.

[12] Orata, P T., *op cit*

situations in which they were acquired to other situations. Systems of general ideas illuminate and clarify human experiences by raising them to the level of abstract, generalized, conceptual understanding."[13]

This statement should give the teacher of mathematics a firm conviction that the mathematics of the secondary school, especially that of the later years, can be of the greatest possible use in achieving the highest aim of education. But again, it is the teacher's obligation to teach so as to assist the pupil in forming those general ideas and attaining the conceptual understanding alluded to in the quotation above.

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1. Betz, William. "The Transfer of Training, with Particular Reference to Geometry." *Fifth Yearbook of the National Council of Teachers of Mathematics*, 1930.
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3. Hedrick, E. R. "Teaching for Transfer." *The Mathematics Teacher*, Vol. XXX, 1937.
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5. Overman, J. R. *An Experimental Study of Certain Factors Affecting Transfer of Training in Arithmetic.* Warwick and York, Baltimore, 1931.

[13] Judd, C. H. *Education as Cultivation of Higher Mental Processes*, p 200 The Macmillan Co., New York, 1936

APPENDIX III

TERMS, SYMBOLS, AND ABBREVIATIONS IN ELEMENTARY MATHEMATICS[1]

". . . disclosing the fourth important fact, that the ratio of the diameter and circumference is as five fourths to four."
HOUSE BILL 246, INDIANA, 1897. (FAILED)

VOCABULARY

A PART at least of the difficulty that pupils find in mathematics arises from the abstractness of some of its conceptions. In past years an exaggerated emphasis upon techniques and skills often failed to reveal the vagueness that enveloped the fundamental concepts of the subject in the minds of individual pupils. If understanding and power are to be explicit aims in mathematical education, then it is essential to emphasize the meaning of all basic terms Mere verbal memorizing of definitions may become a formalistic rote devoid of value in enlarging the pupil's intellectual horizon.

The Commission urges that time and effort be devoted directly to the problem of clarifying the pupil's understanding and increasing his power to use appropriately the technical vocabulary of the subject. The teacher should take special precaution to see that mere imitative usage does not pass for real understanding It recommends further that enlargement of vocabulary be a gradual and continued process, with effective evaluation tests from time to time Each new term should be made the object of study rather than be allowed to enter casually into the working vocabulary of the student.

Terms may be appropriately divided into two categories with respect to the nature of the pupil's desirable mastery over them. A large number of the technical terms belong in the category of terms to be recognized Such recognition involves, first, that the pupil learn which term among several terms offered is the one appropriate in a given situation Second, after this first stage is mastered, it requires that the pupil become proficient in illustrating terms as they are mentioned, for instance, he should be able to sketch figures, or

[1] An extension and revision of the recommendations found in the Report of the National Committee of 1933.

give examples. For the numerous terms in this first category, formal definitions may eventually be presented but need not be made the object of memorization. Formal definitions, despite their logical priority, must be recognized as psychologically furnishing the summary or final synthesis of the process of understanding. Explanations, with simple illustrations, with comparisons and contrasts, serve to carry the pupil over the first difficult stage of acquiring a new concept and placing it in proper relation to his previously acquired system of concepts.

A second category, much more limited than that just considered, comprises terms of so clearly derivative a nature as to admit of definition in terms natural to the pupil. For this smaller category the pupils may reasonably be expected to engage in the problem of devising adequate definitions. This endeavor should be a task for the group, proposals and criticisms being made by all. After a class has reached a workable definition, a standard official definition may be offered by the teacher as an example of what others have done. For all terms in this second category the pupils should feel that they have the power to frame appropriate definitions and pick out serious flaws in defective proposals.

The question whether a term belongs in the first category or in the second cannot be settled offhand or permanently and simultaneously for all classes. The decision must depend upon the intellectual maturity of the pupils, the time available, the extensiveness of the course, and so on Although some thousand technical words are likely to be met in elementary mathematical instruction, a large part of them arise in the field of application.

The Commission urges that teachers recognize in their teaching the futility of pretending to define all terms. That mathematics must be based on certain undefined terms is of course readily seen by those who examine the nature of postulational systems. This logical impossibility can be recognized in part at least by school pupils. It is distinguishable from the psychological complication of defining complex and difficult notions. At an elementary level the logical game of attempting to reduce the list of undefined terms to a minimum or of basing mathematics upon general logical rudiments need not be discussed It is entirely fitting that a rather extended list of terms be accepted without attempt at definition, either initially or later. Among such terms are *space, point, straight line, distance,* and others listed later.

The teacher should be versed in certain points of view with regard to the nature of mathematics, not for the purpose of training the pupil in pointing out logical, psychological, and metaphysical distinctions but rather that the teacher himself may avoid making

wrong statements or even adopting dogmatic attitudes. The distinction between the view of classical antiquity as to the nature of postulates and axioms and the modern conception should be clear to him. For Euclid and his followers through centuries there seemed to be truths so self-evident and basic that one who could not or would not acknowledge their truth would be intellectually or psychologically unfitted to pursue the study of mathematics. The fundamental terms, while undefined or at most partially described, were regarded as having clear and unambiguous meaning. For modern postulational treatment some basic undefined terms are to be thought of as essentially undefined logical parameters capable of interpretations and exemplifications different for each student of the subject. Where consistency is the only requirement, broad generality is available and a system of logical inference becomes at once applicable to realms not anticipated in the original formulation. The pupil may be spared all this sort of discussion. But the teacher at least should realize that to ask whether a "point" is a spot located in the physico-spatial realm, or a triad of numerical coordinates, is to ask a non-pertinent question. Relationships among points rather than the nature of points constitute the concern of formal deductive geometry. For informal geometry one intends to think about the physical space of common sense and daily perception, a space illustrating only approximately the idealized, generalized, and abstract space of pure deductive reasoning. Hence in informal space study it is appropriate to explain, as clearly as possible by comparison, contrast, analogy, and illustrative instance, what is meant by such terms as point and straight line even though actual realization is foreign to the conduct of deductive proof.

In a limited measure the pupils will readily sense an element of arbitrary convention in dealing with basic terms and assumptions. They are playing a game with points, lines, etc., according to rules which usually at first seem vague or tacit, but which the teacher seems to regard as important and directive. They have played other games, learning the rules in most cases incidentally in the process of active and vigorous participation. Most school work is a mildly exciting (or depressing) game, in which the teacher seems to make the rules and act as judge. Language, penmanship, spelling, and arithmetic become collective ventures in which basic principles, if any, may be unsuspected but in which teachers hope to maintain certain standards of accuracy, clarity, neatness, speed, etc. In outdoor sports and games, also, the individual player is not expected to dictate his own rules. The first exercises in formal demonstration offer for most pupils an experience of helplessness in an entirely novel direction. It might be well to emphasize more than has usually been the prac-

tice that the rules of geometry are the result of a consensus of opinion involving many wise and skillful players and developed without essential change in plan over many centuries, and accepted in all countries. But obviously, other games perhaps closely similar are possible.

The Commission believes that it is unnecessary to furnish anything like an "Authorized List" of fundamental terms, since such a list might be in danger of interpretation as both prescriptive and restrictive. The sets of terms that appear in current textbooks are in large part neither too extensive nor too narrow for pedagogical purposes.

It is to be noted that many words of a geometric or a quantitative significance occur in school subjects other than mathematics, or are found in general reading and conversation. In some cases the words are explicitly taught in mathematical works, instances being *cone, convex, spiral.* In other cases the words are related to mathematical terms, instances being *tube, oval, orbit,* which can be discussed in connection with *cylinder, ellipse, circle.* On the other hand there are many words that have a quantitative significance, though not technically mathematical, instances being the words *few* and *several.* The mathematics teacher should not only be familiar with the precise vocabulary of the subject, but also be able to use the less technical quantitative words of daily life in the way best approved

There are a few terms which are of such frequent use and fundamental character, and which at the same time present such wide divergence in usage, according to the tastes and traditions of individual authors, that some special comments seem to be called for. The list of terms selected for these detailed recommendations is to be considered typical rather than exhaustive. A general comment may be made with respect to terms in elementary geometry The powers and also the limitations of algebraic language in describing space figures have motivated a profound change in the fundamental concepts of elementary geometry, a change hardly suggested by the vocabulary itself. Coordinate geometry is well adapted to handle algebraic curves, but relatively poorly adapted to represent many of the figures studied since classical antiquity. With little change in terminology and but minor verbal modifications in the formulation of theorems, the geometry of the Greeks is being transformed into a geometry adapted to algebraic treatment In general, portions of a complete algebraic locus have lost the spotlight in favor of algebraically complete loci. Specialized restrictions, expressible by inequalities, have been waived, with the result of including rather than excluding specialized cases. There is a growing tendency to recognize directed quantities in geometry, a tendency favored by the

conventions in trigonometry. The Commission approves of the increasing use not only of negative angles, but also of negative areas, particularly when this permits a single formulation to cover cases otherwise treated as quite distinct. Such a change in point of view is in no sense a judgment of scientific error on the part of the founders of the subject. It is merely an acknowledgment of the broader points of view made possible by the discoveries and inventions of Fermat, Descartes, and their illustrious successors. (These comments are not intended as any reflection upon the permanent beauty, conciseness, and suggestiveness of a neat synthetic proof, when one can be found. Analytic methods are often almost impracticable because of tedious complexities But a terminology adapted to analytic methods offers rich advantages with no corresponding difficulties beyond a possible break with the past.)

For the sake of unifying mathematics, and avoiding artificial distinction requiring the use of specialized terminology that may seem to the average pupil to be unnecessary or somewhat stilted, good psychological practice tends to carry over into geometrical discussion available algebraic notions rather than to preserve a separate system of terms derived from the Greeks whose knowledge of algebra was at best quite meager. For example, the special terms once common in discussing proportion, such as *antecedent, consequent,* and symbols such as : and the sign : for ratio seem inappropriate for present-day elementary instruction.

There are also trends of current style whose origin may not lie wholly in logic but which the teacher would do well to respect, and symbols used in other classes or outside of school contacts may sometimes be employed Some special recommendations follow.

Geometry

Undefined Terms. The Commission recommends that no attempt be made to define formally terms of logic and geometry, such as *angle, correspondence, direction, distance, figure, magnitude, number, plane, relation, solid, space, straight line, surface, variable,* although the significance of such terms should be made clear by informal explanations and discussions.

Definite Usage Recommended It is the opinion of the Commission that the following general usage is desirable·

1 Line should mean *complete straight line* It has no finite length

2. *Circle* should be considered as the curve.

3. Polygon (including *triangle, square, parallelogram,* and the like) should be considered, by analogy to a circle, as a closed broken line. Similarly, *segment of a circle* should be defined as the figure consisting of a chord and either of its arcs

4. Area of a circle should be defined as the area (expressed in units of area) of the portion of the plane enclosed by the circle. *Area of a polygon* should be treated in the same way.

5. The usage recommended above with respect to plane figures is also recommended with respect to three-dimensional figures. For example, a *sphere* should be regarded as a surface, its *volume* should be defined in a manner analogous to the area of a circle. A similar usage should be followed with respect to such terms as *polyhedron, cone,* and *cylinder.*

6. Circumference should be considered not as a set of points, but as the length (expressed in units of length) of the circle (line) Similarly, *perimeter* should be defined as the length of the broken line which forms a polygon; that is, as the sum of the lengths of the sides

7. Obtuse angle should be defined as an angle greater than a right angle and less than a straight angle and therefore should not be defined merely as an angle greater than a right angle.

8. Convex (as opposed to *concave*), because of its wide application in science and the arts, may well be used and illustrated

9. The term *right triangle* should be preferred to "right angled triangle," this usage being now so well standardized in this country that it may properly be continued in spite of the fact that it is not international. Similarly for *acute triangle, obtuse triangle,* and *oblique triangle.*

10. Such English plurals as *formulas* and *polyhedrons* should be used in place of the Latin and Greek plurals

11. It is not necessary or desirable to distinguish between *axiom* and *postulate.* The word *assumption* can be used in place of these words and may have more meaning to the pupil.

Terms Made General It is the recommendation of the Commission that the modern tendency of having terms made as general as possible should be followed, although the special cases may continue to suggest separate discussion. For example,

1. Isosceles triangle should be defined as a triangle having two equal sides. There should be no limitation of two and only two equal sides.

2 Rectangle should be considered as including a square as a special case, although the notions of a square may be discussed first.

3 Parallelogram should be considered as including a rectangle, and hence a square, as a special case.

4 Trapezoid seems to be most frequently defined in textbooks as a quadrilateral with two and only two sides parallel, but in some cases the "only two" restriction is omitted (which causes a parallelogram to be a special case of a trapezoid) Consistently with the first definition, a *trapezium* is defined as a quadrilateral of which no two

sides are parallel. Some American dictionaries agree with such definitions, preserving the "only two" restriction in the trapezoid; but other dictionaries interchange the two definitions. The Commission recommends that *trapezoid* be defined as first mentioned, and that the word *trapezium* be dropped as unnecessary.

5. Segment (in contrast to *line*) should be used to designate the part of a straight line included between two of its points (as well as the figure formed by an arc of a circle and its chord), this being the usage generally recognized by modern writers. (One obtains the length of a segment, never of a line, since a straight line extends indefinitely in each of two opposite directions.)

Terms to Be Abandoned. It is the opinion of the Commission that the following terms are not of enough usefulness in elementary mathematics at the present time to make their recognition desirable in examinations, and that they serve chiefly to increase the technical vocabulary to the point of being burdensome and unnecessary.

1. Antecedent and consequent.

2 *Equivalent.* This is an unnecessary substitute for the more precise expressions "equal in area" and "equal in volume," or (where no confusion is likely to arise) for the single word "equal."

3. Trapezium.

4. Scholium, oblong, scalene triangle, sect perigon, rhomboid (the term *oblique parallelogram* being sufficient), and *reflex angle* (in elementary geometry).

5 Subtend (in certain connections). Probably pupils have been confused by the statement that an arc subtends a chord or an angle. In such connections the word *has* can be used. There seems, however, to be no adequate simple substitute for the word in such a statement as "A flagpole on top of a building subtends an angle of 10° at a point, etc."

6. Intercept. Since this term has special meaning in higher mathematics, it seems advisable to avoid it in elementary work.

Algebra

1. With respect to equations, the Commission calls attention to the fact that the classification according to degree is comparatively recent and that this probably accounts for the fact that the terminology is so unsettled. The Anglo-American custom of designating an equation of the first degree as a *simple equation* has never been satisfactory, because the term has no real significance The most nearly international terms are *equation of the first degree* (or *first degree equation*) and *linear equation.* The frequent use of both phrases will enlarge the pupil's understanding, though it is to be

noted that the appropriateness of the name *linear equation* is not seen until the graph of such an equation is considered.

2. The term *quadratic equation* is well established. There should be, however, a clear understanding of the somewhat confusing etymology involved, and the use of the description *second degree equation* should be neither prescribed nor banned. The terms *pure quadratic*, *affected quadratic*, *complete quadratic*, and *incomplete quadratic* are fortunately disappearing.

3. As to other special terms, the Commission recommends that the use of the following be avoided in elementary instruction: *evolution* as a general description for finding roots; *involution* for finding powers, *extract* for finding a root, *multiply an equation*, *clear of fractions*, *cancel* and *transpose*, at least until the significance of the terms is entirely clear; *aliquot part* (except in commercial work).

4. The Commission also advises the use of either *system of equations* or *set of equations* rather than the phrase "simultaneous equations."

5. The term *simplify* should not be used in cases where there is possibility of misunderstanding. For purposes of computation, for example, the form $\sqrt{8}$ may be simpler than the form $2\sqrt{2}$, and in some cases it may be better to express $\sqrt{3/4}$ as $\sqrt{0.75}$ instead of as $\frac{1}{2}\sqrt{3}$. In such cases, it is better to give more explicit instructions than to use the misleading term "simplify."

6. The Commission suggests that the word *surd* need not be used for the expression *irrational number*. It recognizes the difficulty generally met by young pupils in distinguishing between *coefficient* and *exponent*, but it feels that it is undesirable to attempt to change these terms which have come to have a standardized meaning and which are reasonably simple. Considerations of a similar nature will probably lead to the retention of such terms as *rationalize* and *extraneous root*.

7. Since the word *plus* is associated with addition and *minus* with subtraction, the Commission feels that "positive" and "negative" numbers are to be preferred to "plus" and "minus" numbers. The fact that "plus" and "minus" serve also to characterize results of rounding off approximate numbers as in 17.2+, and 3.4 — militates against the use of "plus number" for "positive number."

Arithmetic

1. While it is rarely wise to attempt to abandon suddenly the use of words that are well established in our language, the Commission feels called upon to express regret that very young pupils, often in the primary grades, are still required to use such terms as *subtrahend, addend, minuend,* and *multiplicand*. Teachers seem rarely

to understand the real significance of these words, nor do they recognize that they are comparatively modern additions to what used to be a much simpler vocabulary in arithmetic. The Commission recommends that such terms of theoretical arithmetic be used, if at all, only after the sixth grade.

2. Owing to the uncertainty attached to such expressions as "to three decimal places," "to thousandths," "correct to three places," "correct to the nearest thousandth," usage of the following kind is recommended: "To three decimal places" and "to thousandths" should be regarded as identical and as referring to a result carried only to thousandths without considering the figure of ten-thousandths. "*Correct* to three decimal places" and "*correct* to the nearest thousandth" should be regarded as identical and as referring to a result which has been or might have been carried to ten-thousandths and then rounded off to show the nearest thousandth. It may be well to use "exact" in connection with the number of correct decimal places, using "precise" as described below.

3. Because of the appearance in many textbooks of careless definitions of "significant figures," the Commission suggests the following convention: When every figure in a number except the last is exactly correct and the error in the last is less than one-half unit, all the figures have actual meaning and are called significant figures. Zeros are significant when they come within this definition. When they are used simply to give place-value to other digits, zeros are not significant. It may be well to use "precise" in connection with the number of correct significant figures. For example, a measurement may be precise to one part in ten thousand.

4. The Commission recommends the use of the expression "numbers in standard form." By this is meant a numeral with the decimal point located after the first digit and this numeral multiplied, if necessary, by a power of ten to compensate for the change of position of the decimal point, such as $97000 = 9.7 \times 10^4$, $.00097 = 9.7 \times 10^{-4}$, $97^8 = 7.8 \times 10^{15}$, $1/987 = 1.013 \times 10^{-3}$.

SYMBOLS

Mathematics is characteristically symbolic The importance of the general question of symbolism is well recognized—words are themselves symbols. For this section, however, we are interested in nonverbal symbols only Many questions arise concerning the responsibility of those who would originate new symbols. Mathematicians at advanced levels vary greatly in their readiness to propose and use novel symbols There are involved questions of typographical availability as well as questions of intelligibility and good taste. There can be no doubt that new symbols are being continually in-

troduced. Many of them attain favor with the specialists or general mathematical public; more are eventually dismissed as personal idiosyncrasies. At the secondary school level, caution is usually to be preferred to unrestrained originality. One should hesitate to assume that new notation is called for extensively in the traditional branches of elementary mathematics Symbolism that is misleading or merely unintelligible defeats its own purposes, and reasonable uniformity is demanded for maximum service. Consequently the Commission makes certain detailed suggestions regarding individual symbols and even goes so far as to propose a list from which teachers may select such as seem appropriate. Pupils may be encouraged to discuss their need for new symbols and devise as exercise material several symbols to be used in class. But they should ever keep in mind that symbols are not merely for the convenience of the writer. Unless the reader can be expected to be familiar with them, they fail to serve the first objective of communication. For habitual use in examinations or elsewhere, it is wise to be conservative and to use symbols familiar and acceptable to the public generally. From time to time mathematics teachers may be expected to extend approval to new symbols tried and accepted by experimental classes.

Pupils should not be allowed to form the habit of using unusual combinations of symbols and abbreviations. A teacher may well hesitate to approve such writing as the following: "2 boys bt 2 doz. eggs @ 45¢/doz. and 3″ @ 47¢/doz." However, on tests in which speed is being tested one may permit symbols and abbreviations as a sort of shorthand, provided that no ambiguity results

Elementary mathematical symbols fall roughly into four types: (*1*) General literary or typographical symbols, such as ? ″ : , (2) those used throughout mathematics generally, such as the Arabic numerals, $+$, $-$, $=$, $(\,)$, y, x, and the decimal point; (*3*) those used extensively in special parts of the theory but chiefly in informal classroom work; (*4*) standard commercial symbols that the student of mathematics encounters only in applications to commercial problems, such as $, @, # (for pound or for number).

Geometry

1. The symbols $\triangle$ and $\bigcirc$ for triangle and circle are international, although used more extensively in the United States than in other countries. Their use, with appropriate plurals, is recommended.

2 The symbol $\perp$, representing the term *perpendicular* or the phrase *is perpendicular to*, is fairly international and the meaning is apparent Its use is therefore recommended.

3 The symbol $\parallel$ for *parallel* or *is parallel to* is fairly international and is recommended

4. The symbol $\sim$ for *similar* or *is similar to* is international and is recommended.

5. The symbols $\cong$ and $\equiv$ for *congruent* or *is congruent to* have considerable use in this country. The Commission feels that the former, which is fairly international, is to be preferred (in geometry) because it is the more distinctive and suggestive, particularly since $\equiv$ has been widely used for identity.

6. The symbol $\angle$ for *angle* is coming to be generally preferred to any other because of its simplicity, and is therefore recommended.

7. Mathematical writers have not developed any symbols that have general acceptance for the following words, and there seems to be no necessity for making the attempt: *square, rectangle, parallelogram, trapezoid, quadrilateral, semicircle.* (Certain obvious symbols suitable in print, would produce confusion if written carelessly.)

8. The symbol $\widehat{AB}$ for *arc AB* cannot be called international. While the value of the symbol $\frown$ in place of the short word *arc* is doubtful, the Commission sees no objection to its use except possible difficulty in typesetting for publication.

9. The symbol $\therefore$ for *therefore* has a value that is widely recognized, but the symbol $\because$ for *since* is used so seldom that it should be abandoned

10. With respect to the lettering of figures, the Commission calls attention, for purposes of general information, to a convenient method, found in certain European and in some American textbooks, of lettering triangles thus: Capitals represent the vertices, corresponding small letters represent opposite sides, corresponding small Greek letters represent angles, and the primed letters represent the corresponding parts of a congruent or similar triangle. There is much merit in the plan, and the Commission is prepared to recommend it, with optional use of the Greek letters.

11. In general, it is recommended that a single letter be used to designate any geometric magnitude, whenever there is no danger of ambiguity. The use of numbers alone to designate non-numerical magnitudes should be avoided by the use of subscripts or accents,[2] as in A_1, A_2, a', b''.

12. With respect to the symbolism for limits, the Commission calls attention to the fact that the symbol $\doteq$ (for "tends to") is both international and expressive and has constantly grown in favor in recent years Although the subject of limits is not generally treated scientifically in the secondary school, the idea is often mentioned in geometry and a symbol may occasionally be needed

[2] Elementary mathematical notation is relatively simple, and superscripts, which can be easily confused with exponents, are hardly needed There is not the same danger in primed letters

13. The Commission sees no objection to the use of the following abbreviations after the pupil has acquired facility in stating the theorems:

sss for "three sides,"

ass for "two sides and an angle adjacent to one of them,"

sas for "two sides and the included angle," and

asa for "two angles and the included side."

The words *coplanar*, *collinear*, and *concurrent* have now acquired wide use and should be retained. Concerning the terms *ray* and *half-line* the Commission wishes to be non committal.

Algebra

The symbols in elementary algebra are now so well standardized as to require but few comments in a report of this kind. The Commission believes that it is desirable, however, to call attention to the following points.

1. Owing to the frequent use of the letter x, it is preferable in most cases to use the center dot (a raised period) for multiplication in the few cases in which any symbol is necessary. For example, in a case like $1 \cdot 2 \cdot 3 \cdots (x - 1) \cdot x$, the center dot is preferable to the symbol $\times$, while in cases like $2a(x - a)$ no symbol is necessary. The Commission recognizes that the period (as in $a \,.\, b$) is more nearly international than the center dot (as in $a \cdot b$); but since the period will continue to be used in this country as a decimal point, it is likely to cause confusion, to elementary pupils at least, to attempt to use it also as a symbol for multiplication. As noted earlier one writes (in standard form) $97^8 = 7.8 \times 10^{15}$, to avoid possible confusion.

2. In recent years the "decimal" point has acquired new uses. For example, banks quote 95.17 for a Liberty Bond, meaning 95 17/32, and educational psychologists state a pupil's mental age as 12.3, meaning twelve years and three months. The Commission looks with disfavor upon such practices but admits their convenience. It is necessary to inform pupils of these practices.

3. With respect to division, the symbol $\div$ is purely Anglo-American, the symbol : serving in most countries for division as well as ratio. Since neither symbol plays any large part in business life, it seems proper to consider only the needs of algebra, and to make more use of the fractional form and (where the meaning is clear) of the symbol /, and to drop the symbol $\div$ in writing algebraic expressions. In the case of unusually long expressions for numerator and denominator, one may write, "A/B where $A = \ldots$, $B = \ldots$," or otherwise make clear the occurrence of division.

4. With respect to the distinction between the use of $+$ and $-$ as symbols of operation and as symbols to distinguish positive and nega-

tive numbers, the Commission sees no reason for attempting to use smaller signs for the latter purpose, such an attempt never having received international recognition, and there is not sufficient need of two sets of symbols to warrant violating international usage and placing an additional burden on the pupil.

5. In connection with signs it is well to refer to the pedagogical question of explaining the rule of signs in products. One cannot explain, for instance, that a positive number multiplied by a negative number gives a negative number, by saying that subtraction is involved. A correct explanation based upon the distributive law is simple, and throws much light upon the nature of algebra. The proof reveals that algebra resembles a game with the distributive law prescribed, and shows that important consequences can be derived from the law. Special cases can be used to advantage; for example, by applying the distributive law to the product $5(9 + (-3))$, which is known to be 30, it is readily found that $5(-3)$ must be -15. It is better not to attempt to prove the law of signs than to use fallacious reasoning that proves nothing and either confuses or misleads the pupil.[8]

6 With respect to the distinction between the symbols $\equiv$ and $=$ as representing respectively identity and equality, the Commission calls attention to the fact that, while the distinction is generally recognized, the consistent use of the symbols is rarely seen in practice. The Commission recommends that the symbol $\equiv$ be not employed in examinations for the purpose of indicating identity. The teacher, however, should use both symbols if desired.

7 The Commission calls attention to and approves the use of the symbol $\approx$ for *approximately equal to*, the symbol having been introduced first by the engineering societies and having found wide use.

For example: $\dfrac{639 \times 47}{32} \approx \dfrac{600 \times 50}{30} = 1{,}000$

8. With respect to the root sign $\sqrt{}$, the Commission recognizes that convenience of writing assures its continued use in many cases instead of the fractional exponent. It is recommended, however, that in algebraic work involving complicated cases the fractional exponent be used. It should be emphasized that $\sqrt{a}$ (where a is a

[8] For somewhat more advanced pupils it is a good exercise not to use the plus and minus signs to distinguish positive and negative numbers, but to use colors, writing, for example, positive numbers in blue and negative numbers in red A brief addition table should then be made out (in rectangular form), restricted to integers, in which numbers of both colors occur Then a multiplication table should be started. Here the entries for the product of a blue number by a blue number are easily inserted, and the table can be completed by using the distributive law and the addition table. A pupil will understand that it makes no sense to say that a blue number multiplied by a red number means subtraction

positive number) means only the positive root. To have ambiguity in such a symbol would be confusing, and good usage is clearly established. If the root sign itself indicated two numbers, it would not be necessary to use the sign of ambiguity, $\pm$, in the familiar formula for the quadratic equation, but that sign is always printed. Thus $\sqrt{4} = 2$, and it is not good usage to write $\sqrt{4} = \pm 2$. Similarly the two square roots of 3 are $\sqrt{3}$ and $-\sqrt{3}$. Analogous remarks can be made about $a^{1/2}$, $\sqrt[4]{a}$, $a^{1/4}$, etc.

When imaginaries are used, the symbol i should be employed instead of $\sqrt{-1}$ except possibly in the first presentation of the subject. The reason for writing $\sqrt{-a}$ (where a is positive) in the form $i\sqrt{a}$ becomes apparent upon considering multiplication. The product of $(i\sqrt{6})(i\sqrt{5}) = i^2\sqrt{6}\sqrt{5} = -\sqrt{30}$, which value might not be obtained if the numbers are left in the form $\sqrt{-6}$ and $\sqrt{-5}$, respectively. Finally it is to be noted that neither i nor $-i$ should be referred to as positive, "positive" implies "real."

9. As to the factorial symbols such as in $5!$ and $\underline{5|}$, to represent $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, the tendency is very general to abandon the second one, probably on account of the difficulty of printing it. In such abandonment the Commission concurs. (This question is not, however, of great importance in the general courses in the high school.)

10. With respect to symbols for an unknown number there has been a noteworthy change within recent years. While the Cartesian use of x and y will doubtless continue for two general unknowns, the recognition that the formula is, in the broad use of the term, a central feature of algebra has led in the extended use of the initial letters of words to represent quantities. This is simply illustrated by the formula $A = \pi r^2$. The custom referred to is now international and should be fully recognized in the schools.

11. Unfortunately, it is still necessary to advise abandoning the double colon in proportion, and the symbol $\propto$ in variation, although both of these symbols are practically obsolete in mathematics except in textbooks.

VERBAL ABBREVIATIONS

There is much difference of taste as to how extensively verbal abbreviations should be used and how long a list of abbreviations should be allowed. The pupil should be curbed in any tendency toward the slovenly writing that relies on wholesale use of abbreviations for words when they occur in complete sentences. Abbreviated word symbols are acceptable only in so far as they are easily understood and are called for in the interest of compactness and economy of time.

It may be well for a class to discuss from time to time the words for which abbreviations would be most welcome, and to vote approval, for blackboard work, or upon tests during which a student is pressed for time, of the use of simple abbreviations; but it should be insisted that abbreviations be the same for all users and that no unauthorized abbreviations be allowed. For example, if "hyp" be used for "hypothesis" and "hypn" for "hypotenuse," then to write "hyp" for hypotenuse should be treated as a breach of accepted custom. It is to be hoped that eventually teachers from various schools and from various parts of the country may come to some common official understanding as to what abbreviations seem appropriate, so that students taking examinations in schools other than those in which they were first trained may be free to use appropriate abbreviations without fear of ambiguity. No such widespread agreements have been reached as yet, however.

In closing this appendix it may be well to quote the following forceful injunction from the Report of the National Committee of 1923:

"It is to be feared that many teachers encourage the use of a kind of vulgar mathematical slang when they allow such words as 'tan' and 'cos,' for tangent and cosine, and habitually call their subject by the title 'math.' "

APPENDIX IV

EQUIPMENT OF THE MATHEMATICS CLASSROOM

> *"We [illegible] thankful for it, and entirely ashamed [illegible] good use of it."*
>
> JOHN RUSKIN, SESAME AND LILIES

The appropriate furnishing and equipping of a mathematics classroom is receiving more attention than formerly, many teachers not being contented with four walls and a blackboard. The movement is partly a manifestation of the general wish to remove barrenness from schoolrooms and make them congenial places whose good cheer invites study and reflection. But it is also thought that proper equipment aids instruction, if indeed it is not an essential for the type of teaching that has been made necessary by the present clientele of the schools.

Although the Commission strongly recommends that careful attention be given to equipping the mathematics classroom so as to aid instruction and stimulate interest, it believes that a balanced point of view should be maintained. The argument for equipment is injured rather than helped if reasons are given that are not sound, or if comparisons are made that are not warranted. For example, equipment does not bear the same relation to mathematics that it does to a laboratory science, such as chemistry or physics. Apparatus, sometimes very elaborate and expensive, is needed by the physicist; but the mathematician does his work now as formerly, with paper and pencil except when making extensive numerical computations. For the strongest students in chemistry the most extensive equipment might be needed; for the strongest students in mathematics nothing is usually required but books and a place to work quietly and undisturbed. Classes may, as a matter of fact, be overindulged with even the simple basic instruments of mathematics. From a geometry course a pupil should derive ability to make fair freehand drawings. Both his straight lines and his circles should present a tolerable appearance; but they will not do so if all his drawing is done with ruler and compasses. He will not be able to draw well to scale assisted only by his eye if he has worked exclusively on cross section paper or on a blackboard divided nicely into squares. Since

drawing instruments are not habitually carried, the ability to make free-hand sketches of both plane and space figures should be obvious enough. It can be said also that even the immediate availability of instruments does not indicate that they should unreservedly be used; for example, although the digits on our hands lend themselves to problems in addition, their use for such a purpose is somewhat in disrepute. Unless a teacher is careful, he may forbid a pupil to find the sum of 15 and 12 by using his fingers, only to discover that he obtains their product by using a slide rule.

The place where proper equipment has by tradition been most recognized is in connection with teaching solid geometry and for giving adequate and clear space perception. Stereoscopic views of figures do more than assist in understanding proofs of theorems; they cause lines to stand out so clearly that a figure in a plane acquires some of the charm of a graceful and well proportioned structure. More aids in the way of models and views at a moderate cost are available than formerly, and they should be used to instruct and to arouse interest. Certain basic figures and solids should be prominent in the classroom, to which they give an atmosphere that is appropriate to the subject that is being taught. A knowledge of the properties of figures and a recognition of their importance is stimulated by having the figures form a part of one's habitual surroundings. Equipment appropriate for the physics laboratory may also be employed in the mathematics room. For example, instruments that explain the resolution of forces will assist in understanding some of the applications of trigonometry.

The use of calculating machines in connection with commercial instruction is already well established. It is obvious that pupils should be taught methods used in the business world and that they should be made familiar with appliances that have greatly simplified numerical computation. An understanding of machine calculation may increase a pupil's opportunity of finding employment. Machines also have an important place in mathematical work other than commercial work. The time that they save in the making of necessary computations can be devoted to covering more theory or to studying more thoroughly concepts and principles Indirectly, then, machines, when they are properly utilized, can aid in basic mathematical instruction. Pupils seriously interested in mathematical study, as well as those who desire thorough commercial training, should have the opportunity to gain some skill in machine calculation. Much can be accomplished in the way of instruction by means of the lower priced machines designed especially for school use.

Instruments for surveying, especially a transit, have considerable usefulness in teaching parts of geometry and trigonometry. The interest of many pupils is increased by means of practical problems

and an instrument makes it possible for pupils to secure their own data for a variety of exercises. In addition, skill in the use of instruments is valuable. An accurate engineer's transit is not needed unless work in surveying of a high quality is attempted. Very inexpensive transits designed for the use of builders and for farm surveying, as well as special school instruments, are on the market, and they are suitable for demonstration purposes and for the field work that may go with a course in trigonometry. Such instruments have made it possible for schools to provide a serious unit of field work, thereby giving a type of training formerly seldom available. Sextants for school use are also available at low prices and are instructive instruments. A person familiar with the use of such an instrument may desire one for recreational purposes, and individual ownership of inexpensive instruments is quite possible.

More varied aids than those discussed above are needed for the rank and file of pupils if they are to understand mathematics and appreciate its place in current life. In Chapter VII it was stated that the slower pupils "must handle, measure, cut, count, draw, make models, draw graphs . . . ," in order to learn. Accordingly the necessary equipment must be provided, and it is not expensive. The interest of average pupils and even of some of the stronger ones may be stimulated by gathering, organizing, and preparing posters or other material that will show the wide and varied uses of mathematics. Some attention has already been given to this subject in Chapter V, and the possibilities open to a resourceful teacher are very great. Displays are instructive things in themselves as well as a means of stimulating study. Pictures and diagrams do not need to be restricted to applications of mathematics; theorems and demonstrations lend themselves to the purpose. The theorem of Pythagoras with the customary proof is worthy of display in any mathematics room; one of the simple methods of trisecting an angle by compasses and a straight edge with marks upon it, would make an exhibit that would help correct the current misconception about one of the most famous of all mathematical problems.

Proper furnishing of a room is required for displaying and using equipment. There should be cases for books, for instruments, for models, and wall space for pictures, and posters. Storage space should be conveniently accessible for keeping the work that is done by different classes. There should be a good drawing table, properly equipped, at which careful drawing can be done; and there should be other work tables. In a large school with a number of rooms devoted to mathematics instruction, a separate room to serve as laboratory and museum can be used with very valuable results. A projecting lantern and screen can be used to advantage at times, and the future will probably see the development of more and better films having a mathematical bearing.

Some suggested equipment is listed below.[1] While it is to be expected that an appropriate amount of the school budget will be allotted for the purchase of equipment and supplies for the mathematics department, it is to be remembered that most of the charts and posters, many of the models, and some of the other equipment can be made by the pupils.

INSTRUMENTS

Under this heading should be included not only obvious items, such as compasses, drawing instruments, slide rules, protractors, parallel rulers, a pantograph, a vernier, and a vernier caliper, but the more elaborate equipment, such as a surveying transit, a sextant, and a calculating machine, whose uses and advantages have been previously discussed.[2] A spherical blackboard should also be included, as well as cross section blackboard graph charts.[3]

MODELS

Among models should be found: Prisms,[4] pyramids, cylinders, cones, spheres, the five regular polyhedrons. There should also be a stereoscope and views.

SHOWCASE AND WALL DISPLAYS

An abacus, Napier's bones, sundials, pictures of ancient clocks, pictures or models of ancient surveying instruments, various models made by pupils, drawings and proofs of important theorems, pictures of general mathematical interest, posters, etc., may be among the showcase and wall displays.

SUPPLIES

Supplies should include: Graph paper· rectangular (several different units), logarithmic, semi-logarithmic, polar. Mimeograph ma-

[1] See also Woodring and Sanford, *Enriched Teaching of Mathematics in the Junior and Senior High School*, pp. 104-112.

[2] See Woodring and Sanford, *loc. cit.*, for dealers in instruments Advertisements of mathematical supplies and equipment will also be found in *The Mathematics Teacher* and *School Science and Mathematics* Well known mail-order houses handle low priced transits Among the better known calculating machines are the Monroe, the Marchant, the Frieden, the Mathematon, and the Mercedes.

[3] Different sizes of such charts on slated cloth are available Since they are movable they present certain advantages over a permanent wall board that is ruled.

[4] There should be prisms with different numbers of faces, also oblique prisms and truncated prisms There should be a corresponding variety of pyramids, and cones cut to show the different conic sections

terial General supplies—paper, colored chalk, Bristol board, pasteboard, tracing material

TEACHER'S FILE

In the teacher's file should be: Instructional materials, including applications. Test materials. Bibliographies.

MATHEMATICAL BOOKS AND PERIODICALS FOR THE HIGH SCHOOL LIBRARY

Texts. The library should contain a good selection of standard texts covering elementary algebra, geometry, trigonometry, high school general mathematics, college algebra, analytic geometry, mathematics of finance, elementary statistics, calculus,[5] and unified college mathematics. There should also be an adequate number of good mathematical tables, available in the library and at special places that are designed for work involving calculations. The library should also have standard works on the teaching of mathematics for the use of teachers.

Reference Books. The list of reference books below does not aim at completeness, but it covers very adequately the history of mathematics, and also provides for other types of supplementary reading and instruction.

Abbott, Edwin A. *Flatland.* (2nd edition). Little, Brown and Co., Boston, 1926. 155pp., $1.25.

American Council on Education. *Achievements of Civilization.*

No. 2. *The Story of Numbers.* 32pp., $0.10.

No. 3. *The Story of Weights and Measures.* $0.10.

No. 4. *The Story of Our Calendar.* 32pp., $0.10.

No. 5. *Telling Time Throughout the Centuries.* 64pp., $0.20.

American Council on Education, 744 Jackson Place NW, Washington, D. C., 1933.

Andrews, F. Emerson. *New Numbers.* Harcourt, Brace and Co., New York, 62pp., $0.50.

Andrews, William S. *Magic Squares and Cubes.* Open Court Publishing Co., Chicago, 1908. 199pp. (Out of print).

Archibald, R. C. *Outline of the History of Mathematics.* (3rd edition). Mathematical Association of America, Oberlin, Ohio, 1936. 62pp. $0.50.

[5] Some of the briefer and simpler books on calculus are to be recommended. *Calculus Made Easy,* by Silvanus P. Thompson (Macmillan and Co., 2nd ed., 1919, 265 pp.), is likely to arouse the interest of students, a fact that offsets criticism one might make of a deceptive suggestion of the title and the unconventionality and lack of precision of treatment.

Ball, W. W. R. *Mathematical Recreations and Essays.* (7th edition). Macmillan Co., New York, 1920. 506pp., $3.50.

Bell, Eric. *Men of Mathematics.* Simon and Schuster, New York, 1937. 592pp., $5 00.

Bell, Eric. *Queen of the Sciences.* Williams and Wilkins Co., Baltimore, 1931. 138pp. $1.00.

Bentley, W. A. and Humphreys, W. J. *Snow Crystals.* McGraw-Hill Book Company, New York, 1931. 227pp., $10.00.

Bradley, A. D. *Mathematics of Map Projection and Navigation.* Lafayette Instruments, Inc., New York, 1938. 108pp. (mimeographed), $1.75.

Breslich, Ernst R. *Excursions in Mathematics.* The Orthovis Co., Chicago, 1937. 47pp., $1.50.

Brodetsky, S. *A First Course in Nomography.* Open Court Publishing Co., Chicago, 1925. 160pp., $3.00.

Cajori, F. *History of Mathematics* (2nd edition). Macmillan Co., New York, 1919. 516pp., $4.50.

Candy, Albert L. *Construction, Classification and Census of Magic Squares of Order Five.* The Author, 1003 H Street, Lincoln, Neb., 1938. 141pp., $1.00.

Collins, A. F. *Fun with Figures.* Appleton Century Co, New York, 1933. 253pp., $2.00.

Dantzig, Tobias. *Number, the Language of Science* (3rd edition). Macmillan Co., New York, 1939. 320pp. $3 00.

Dudeney, H. E. *Amusements in Mathematics.* Thomas Nelson and Sons, New York, 1917. 258pp., $1.50.

Dudeney, H. E. *Canterbury Puzzles* Thomas Nelson and Sons, London, 1919. 255pp., $1.50.

Dudeney, H E. *Modern Puzzles and How to Solve Them.* F. A. Stokes and Co., New York, 1926. 190pp , $1 25.

Dudeney, H. E *Puzzles and Curious Problems* Thomas Nelson and Sons, London, 1932. 195pp , $1 50

Heath, R. V. *Mathemagic.* Simon and Schuster, New York, 1933. 138pp., $1.75.

Hogben, Lancelot. *Mathematics for the Million* W. W Norton Co., New York, 1937. 647pp., $3.75.

Hopper, V. F. *Medieval Number Symbolism.* Columbia University Press, New York, 1938. 241pp., $2.80

Hornung, C. P *Handbook of Designs and Devices* Harper & Brothers, New York, 1932 204pp , $2.50

Jones, S. I. *Mathematical Nuts.* The Author, Life and Casualty Bldg , Nashville, Tenn , 1932 340pp , $3 50

Jones, S. I. *Mathematical Wrinkles* The Author, Life and Casualty Bldg , Nashville, Tenn , 1926 (3rd edition). 361pp., $3 00

Karpinski, L. C. *The History of Arithmetic.* Rand McNally and Co., Chicago, 1935. 200pp., $2.00.

Levinson, Horace C. *Your Chance to Win: The Laws of Chance and Probability.* Farrar and Rinehart, New York, 1939. 343pp., $2.50.

Licks, H. E. *Recreations in Mathematics.* D. Van Nostrand Co., New York, 1917. 155pp., $1.50.

Loomis, E. S. *Pythagorean Theorem.* Masons, 1321 W. 111th St., Cleveland, Ohio, 1927, 214pp.

MacMahon, P. A. *New Mathematical Pastimes.* Macmillan Co., New York, 1930. 114pp., $4.00.

McKay, Herbert. *Adventures in Arithmetic.* Macmillan Co., New York, 1939. $1.50.

Merrill, Helen A. *Mathematical Excursions.* Bruce Humphries, Inc., Boston, Mass., 1933. 145pp., $2.00.

Sanford, Vera. *Short History of Mathematics.* Houghton Mifflin Co., New York, 1930. 400pp., $3.25.

Shuster, C. and Bedford, F. *Field Work in Mathematics.* American Book Co., New York, 1935. 168pp., $1.20.

Smith, D. E. *History of Mathematics.* Ginn and Co., Boston, 1923, 25. 2v. 1321pp., $5.00 each.

Smith, D. E. *Mathematics, Our Debt to Greece and Rome.* Marshall Jones Co., Boston. 175pp., $1.75.

Smith, D. E. *Number Stories of Long Ago.* Ginn and Co., Boston, 1919. 136pp., $0.68.

Smith, D. E. and Ginsburg, J. *Numbers and Numerals.* Bureau of Publications, Teachers College, Columbia University, New York, 1937. 52pp., $0.35.

Steinhaus, Hugo. *Mathematical Snapshots.* G. E. Stechert and Co., New York, 1938. 135pp., $2.50.

Weeks, Raymond. *Boys' Own Arithmetic.* E. P. Dutton and Co., New York, 1924. 188pp., $2.00.

White, W. F. *A Scrapbook of Elementary Mathematics.* Open Court Publishing Co., Chicago, 1927. 248pp., $1.50.

Woodring, Maxie N. and Sanford, Vera. *Enriched Teaching of Mathematics in the Junior and Senior High School* (revised edition). Bureau of Publications, Teachers College, Columbia University, New York, 1938. 133pp., $1.75.

The last book, which has been previously referred to in footnotes on earlier pages, should be well known to all teachers of mathematics. Its 133 pages give in convenient form much material and many references brought together in no other place. Various sections in it are especially useful in connection with pages 68-71 of this Report, and with the last three horizontal divisions of the Grade Placement

Chart, which forms Appendix V. Blank pages in the book give a ready place to supplement it by new references that may be found in current literature.

Periodicals. Such magazines as *The Scientific American* and *Popular Science* have a certain amount of mathematical interest at times.

Periodicals of special interest in the secondary mathematics field are:

The Mathematics Teacher. 525 West 120th Street, New York City. Eight copies a year. $2.00 a year.

This is a publication of the National Council of Teachers of Mathematics. A subscription carries membership in the Council. The magazine is devoted to the interests of mathematics in elementary and secondary schools.

School Science and Mathematics. Menasha, Wis. Nine copies a year. $2.50 a year.

This is a publication of the Central Association of Science and Mathematics Teachers. Contains articles dealing with mathematics and its teaching, and also a problem department.

In addition to *The Mathematics Teacher,* the National Council has, since 1926, published annually a *Yearbook.* These works have covered a wide variety of articles and studies valuable to elementary and secondary mathematics teachers.

APPENDIX

GRADE PLACEMENT CHART FOR

	GRADE 7	GRADE 8	GRADE 9
ARITHMETIC (Number and Computation)	1. Experiences, language, and ideas 2. Fundamental processes with whole numbers, fractions, and decimals, and related principles 3. Significant applications. 4 (Optional) Elementary approximate computation	1. Experiences, language, and ideas (continued) 2 Fundamental processes and related principles (reviewed and extended). 3 Significant applications. 4 (Optional) Approximate computation (continued)	1. Review and extension of concepts and skills 2. Applications, preferably in connection with algebra 3. (Optional) Logarithms and the slide rule.
GEOMETRY (Space Perception, Demonstration)	*(Informal)* 1. Experiences, language, and ideas 2. Drawing or constructing basic figures. 3 Direct measurement (lengths and angles). 4 Indirect measurement (areas and volumes). 5 (Optional) Application of elementary approximate computation 6. Related facts and principles 7 Significant applications.	*(Informal)* 1 Experiences, language, and ideas, (continued) 2 Drawing or constructing important figures 3 Indirect measurement 4. (Optional) Application of elementary approximate computation (continued) 5 Related facts and principles (continued) 6 Significant applications	*(Informal)* 1 Review and extension of concepts, skills, facts, and relations 2. Applications, preferably in connection with algebra 3. (Optional) Introduction to demonstrative geometry. (In grades 9 and 10, algebra and geometry may be closely correlated)
GRAPHIC REPRESENTATION	1 Interpretation of simple pictograms or statistical graphs 2. Graphic representation of simple statistical data	1 Interpretation of statistical graphs 2 Graphic representation of everyday statistical data (bar graph, line graph, circle graph) 3 (Optional) Tabular and graphic representation of relationships expressed by simple formulas	1. Statistical graphs (reviewed and extended). 2. Functional graphs (formulas, $y=ax+b$, $y=ax^2$).

NOTE 1. The central theme or core of each year's technical work is indicated by means of
NOTE 2 In general, no single class should attempt all the optional lines of work or types of omitted if local conditions require such modification This is especially true of certain of the

V

PLAN OF CHAPTER V, GRADES 7–12

GRADE 10	GRADE 11	GRADE 12
1 Review and extension, preferably in connection with applied problems. 2 The use of irrational numbers 3 Approximate computation 4. (Optional) The use of the slide rule	1. Review and extension. 2. Study of the number system. 3 Approximate computation 4. (Optional) Study and use of the slide rule. 5 (Optional) Use of calculating machines.	1. Review and extension. 2 The number system (complex numbers). 3 Approximate computation, including use of the derivative.
(Formal) 1 Transition to formal geometry. 2. Acquisition of skill in demonstration. 3 Familiarity with facts and propositions, properly organized 4. Development of elementary spatial insight.	1 Review in connection with trigonometry	1. Basic propositions in solid geometry with properties and mensuration of solids 2. Equations of straight line and circle systematically studied 3. Simple locus problems 4 (Optional) Introduction to parabola and ellipse
1. Review and extension, preferably in connection with the social studies and science programs 2 (Optional) Graphs of simple equations	1 Representation of more complicated statistical data 2 Graphs of linear and quadratic functions 3 Graphic solutions of problems 4 Graphs of trigonometric functions	1 Graphic solution of equations 2 Representation of complex numbers (either in rectangular or in polar coordinates) 3 (Optional) Use of logarithmic paper

double borders Mathematical modes of thinking, etc, should be stressed in all years
enrichment suggested for each year Some of the topics not marked optional can be deferred or
historical topics suggested

GRADE PLACEMENT

	GRADE 7	GRADE 8	GRADE 9
ALGEBRA	1. (Optional) The use and application of formulas as expressions of simple relations and of fundamental rules of procedure	(*Informal*) 1. The shorthand of algebra (concepts and simple techniques) 2. The formula (evaluation and construction) 3. The equation (simple cases) 4. (Optional) Signed numbers and their uses. 5. Significant applications.	(*Elementary*) 1. Language and ideas (extended) 2. Fundamental techniques. 3. Fundamental principles 4. The functional core of algebra (formula, table, equation, graph). 5. Significant applications (See text for details) (In grades 9 and 10, algebra and geometry may be closely correlated)
TRIGONOMETRY	(*Preparatory Work*) 1. Scale drawing (begun) 2. Measurement of lengths and angles 3. Ratio (begun)	(*Preparatory Work*) 1. Scale drawing (continued) 2. Similarity and proportion. 3. Out-of-door work in indirect measurement 4. The use of simple instruments	(*Numerical*) 1. Language and ideas. 2. Necessary skills (drawing to scale, using tables of sines, cosines, tangents) 3. Applied problems. 4. Approximate computation arising from use of tables 5. (Optional) The slide rule
MATHEMATICAL MODES OF THINKING, HABITS, ATTITUDES, TYPES OF APPRECIATION	1. The development of habits of correctness in computation, measurement, and drawing, and in making verbal statements. 2. The development of habits of estimating and of checking 3. Learning to interpret and to analyze elementary problem situations 4. Learning to prepare neatly and economically arranged written solutions of suitable mathematical problems. 5. The development of an interest in the study of simple quantitative relationships with the aid of the table, the graph, the formula, and the equation 6. Learning to appreciate the place of mathematics in everyday life.		1. Continuation of the modes of thinking outlined for grades 7 and 8. 2. Learning to understand and to apply relational thinking (the idea of dependence, of functional thinking) as a key method of dealing with quantitative changes arising in nature, in business, and in everyday life.

CHART (*Continued*)

GRADE 10	GRADE 11	GRADE 12
1 Use of algebra in connection with geometric proofs and work.	(*Intermediate*) 1. Review and extension of basic concepts and techniques. 2 Linear functions and equations 3. Quadratic functions and equations. 4. Radicals and radical equations 5 Logarithms and the slide rule. 6. Series (arithmetic, geometric, binominal). 7–11 (See text.)	(*Advanced, and Elements of Differential Calculus*) 1. Basic work in the theory of equations, including determination of real roots. 2. Permutations, combinations, and simple work in probability. 3 Differentiation of polynomials 4. Slopes, maxima and minima, rates of change, etc 5 (Optional or as substitutes) Elements of finance, statistics.
(*Numerical*) 1. Review and extension 2. Functions of 30°, 45°, 60° 3 Significant applied problems involving use of natural trigonometric functions. 4 The slide rule 5 (Optional) Use of logarithms and slide rule	(*Formal*) 1 The six trigonometric functions. 2 Basic identities 3. The addition formulas. 4 Double-angle and half-angle formulas 5 Laws of sines, cosines, tangents 6. Solution of triangles 7. Components and resultants 8 Simple identities and equations. 9. Field work	1 Review 2. Radian measure 3 Inverse functions 4. Identities and equations. 5 DeMoivre's Theorem (When desirable, topics may be moved from grade 11 to grade 12, and some topics above may be omitted)
1 Continuation of the modes of thinking outlined for grades 7, 8, and 9. 2 Learning to understand and to apply the deductive type of thinking as a method of dealing with situations involving sets of interdependent concepts and closely related propositions	1 Continuation of the modes of thinking suggested for grades 7–10. 2. A more systematic application of functional and statistical thinking, not only in mathematics, but also in science, in the social studies, in economics, and in related fields 3 The development of greater skill in using deductive reasoning both in mathematics and in life situations. 4 Learning to appreciate more fully both the service values and the cultural values of mathematics	

GRADE PLACEMENT

	GRADE 7	GRADE 8	GRADE 9
HISTORY OF MATHEMATICS	1 The story of numbers and numerals 2. The story of measurement	1. The story of the decimal system and of computation 2. Early history of geometry.	1. The story of algebraic symbolism 2. The story of indirect measurement 3 (Optional) History of signed numbers and elementary aspects of irrational numbers
CORRELATED MATHEMATICAL PROJECTS AND ACTIVITIES	1. Projects (home, school, community) 2. The school bank 3. Simple measurement projects. 4. Simple geometric designs in nature and art. 5 Making mathematical source books and posters. 6. Correlation with centers of interest 7. Mathematical recreations.	1. Banks and banking 2. Income taxes. 3. Family budgets 4. Installment buying 5. Out-of-door work in measuring heights and distances. 6. Making geometric designs or posters 7. Mathematical recreations.	1 The place of mathematics in the modern world. 2. The mathematics of business and of the shop. 3. Graphic devices used in everyday life. 4 Correlation with science and the social studies. 5. Elementary astronomy (descriptive) 6 Mathematical recreations

CHART (*Continued*)

<table>
<tr><th>GRADE 10</th><th>GRADE 11</th><th>GRADE 12</th></tr>
<tr><td>1. The development of geometry in Egypt, Babylonia, and Greece
2 Great Greek mathematicians
3 Pre-Greek mathematics</td><td colspan="2">See Note 2
1 Systematic development of algebra, as centering around the solution of equations, leading contributors
2 Beginning of the modern period, Descartes, Newton, Leibniz.
3. Great development of analysis since 1700, leading contributors.
4 Mathematical physics and astronomy, leading contributors
5. The mathematical discovery of Neptune
6 The discovery of non-Euclidean geometry
7. Development of mathematics in America, the influence of Bowditch, Peirce, etc</td></tr>
<tr><td>1 Using postulational thinking in life situations
2. The geometry of architecture, of surveying, of design, and of related fields
3. Mathematical recreations (fallacies and the like)</td><td>1. Calculating machines
2. Making simple surveying instruments
3 Surveying projects
4. Introduction to astronomy (mathematical)
5. Mathematical recreations</td><td>1 Statistics and modern life.
2 Mathematics of finance
3 Elementary work in mechanics
4 The mathematics of the telescope
5 The mathematics needed in the leading professions
6 Mathematical recreations.</td></tr>
</table>

APPENDIX VI

ONE POSSIBLE SELECTION AND ARRANGEMENT OF INSTRUCTIONAL TOPICS FOR SLOW PUPILS

See page 141

The chart makes provision both for a little very simple algebraic work in grades 7 and 8, and for deferring such study until grade 9

	7TH GRADE	8TH GRADE	9TH GRADE
ARITHMETIC	1 Reteaching of reading and writing numbers, including integers, fractions, and decimals 2 Reteaching of fundamental processes 3 Arithmetic average. 4 Estimation of magnitude, e g, number of objects in a group, lengths, areas 5 Checking results (units and numerical value). 6 Rounding off numbers 7 Denominate numbers time, money, linear measure, square measure, relationships between units 8. Percentage understanding of percentage, with simple applications 9 Simple financial and other everyday problems	1 Review of integers fractions, decimals, and of fundamental processes 2 Continuation of estimation and checking 3 Idea of approximation and simple approximate computations 4 Denominate numbers weight, cubic measure, reading of meters (optional) 5 Percentage more difficult work with simple applications 6 Everyday problems of average difficulty (personal, home, and community) 7 Very simple cases of ratio and proportion	1 Review of integers, fractions, and decimals, and of fundamental operations 2 Continuation of estimating and checking 3 Denominate numbers continued and extended. 4 Percentage more difficult applications 5 Problems related to the shop, and to other school or life situations. 6 Reading and interpreting tables
ALGEBRA	*Optional* Expressing simple relations (such as occur in the study of perimeters, areas, and percentage) by means of formulas.	*Optional* 1 Use of formulas, especially in the study of perimeters, areas, and volumes 2 Use of equations in the solution of simple verbal problems.	1 Substituting numbers in algebraic expressions 2 Addition, subtraction, multiplication, and division of simple monomials. 3 Evaluating easy formulas 4 Use of negative numbers 5 More extensive work with simple types of equations Note If no algebraic work is given in grades 7 and 8, the suggestions above will need to be appropriately modified

GEOMETRY (including graphs)	1 Lines: measuring of segments. naming with one or two letters, classifying as vertical, horizontal, slanting, parallel, perpendicular 2. Circles: concept and simple relation. 3 Angles: reading, classifying, measuring and estimating, drawing with protractor. 4. Triangles. reading and classifying. 5. Two-dimensional figures· perimeter and area of rectangle (including square), triangle, parallelogram. 6 Graphs reading and interpreting pictograms, line graphs, bar graphs, and circle graphs, drawing of very easy bar graphs and line graphs.	1 Two-dimensional figures· continued study of area and perimeter of rectangles, triangles, parallelograms, and the circle 2. Three-dimensional figures. volume of rectangular solid (including cube) and cylinder 3 Constructions circles, bisectors of segments, equal angles, bisectors of angles, perpendiculars, hexagons, octagons, parallels, designs. 4. Scale drawings. 5. Graphs: reading, interpreting, and making more difficult graphs	1. Review of· perimeters and areas of rectangle, triangle, parallelogram, and circle, volumes of rectangular solid and cylinder Areas of rectangular solid and cylinder 2. Constructions: triangles, given *sas*, *asa*, *sss;* triangle similar to given triangle; finding center of circle when an arc is given, tangents to circles, angles of 30°, 45°, 60°. 3 Elementary theorems (without proofs)· sum of angles of a triangle, Pythagorean Theorem, theorems of congruence and similarity. 4. Introduction to indirect measurement, simple numerical work with tangent, sine, and cosine. 5 Graphs drawing circle graphs, reading and constructing more difficult bar and line graphs based on data significant to the pupil
ENRICHMENT MATERIALS (To be introduced at such times in the three years and to such an extent as may be feasible.)	*History of Mathematics* 1. Early need of mathematics 2 Dependence of civilization upon mathematics. *Geometric Form in Nature and Practical Arts* 1 The earth, sun, moon, planets, flowers, snowflakes, honeycomb 2 House furnishings, building, weaving, design, machinery 3 Gothic and modern architecture *Vocational Needs of Mathematics* Information concerning mathematical needs in the trades and vocations which this group of pupils will probably enter.	*Measurement* 1 History of early surveying and the origin of non-standard units of measure still in use, such as hand and pace 2 Information concerning U.S. Bureau of Standards and standard units of measure 3 Knowledge concerning the metric system and actual familiarity with several of the most used metric units 4 Information concerning instruments used for very exact measurement 5 Information concerning such instruments as calipers, verniers, carpenter's square. 6. Information concerning and some use of such instruments as the transit, plane table, pantograph *Instruments for Rapid Calculation* Demonstrated use of slide rule and, if possible, of a calculating machine	*Mathematics in Life Situations* 1 Collecting and interpreting statistical information of general interest. 2 Discovering mathematical terms and ideas used in other school subjects, in texts, and in life situations. 3. Collecting and analyzing problem data of general interest. *Recreations* 1. Simpler cases of mathematical recreations as given in appropriate books 2 Projects such as. a. Laying out an acre. b Measuring a city block. c Making a cubic foot and a cubic decimeter d. Making a mathematical scrapbook.